

# Small $x$ resummation of rapidity distributions: the case of Higgs production

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## Abstract:

We provide a method for the all order computation of small  $x$  contributions at the leading logarithmic level to cross-sections which are differential in rapidity. The method is based on a generalization to rapidity distributions of the high energy (or  $k_T$ ) factorization theorem hitherto proven for inclusive cross-sections. We apply the method to Higgs production in gluon-gluon fusion, both with finite top mass and in the infinite mass limit: in both cases, we determine all-order resummed expressions, as well as explicit expressions for the leading small  $x$  terms up to NNLO. We use our result to construct an explicit approximate analytic expression of the finite-mass NLO rapidity distribution and an estimate of finite-mass corrections at NNLO.

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>High energy factorization of inclusive cross-sections</b>	<b>4</b>
2.1	Factorized cross-section . . . . .	5
2.2	Hard part . . . . .	8
2.3	Ladder part: generalities and double log approximation . . . . .	9
2.4	Ladder part: $LLx$ case . . . . .	14
2.5	Cross-section resummation . . . . .	16
2.6	Hadroproduction and quark-initiated processes . . . . .	17
<b>3</b>	<b>High energy factorization of rapidity distributions</b>	<b>19</b>
3.1	Kinematics and factorization . . . . .	19
3.2	Ladder part and rapidity dependence . . . . .	21
3.3	Resummation of rapidity distributions . . . . .	23
<b>4</b>	<b>Higgs rapidity distribution at small <math>x</math></b>	<b>25</b>
4.1	Heavy top approximation . . . . .	27
4.2	Rapidity distribution with finite $m_{\text{top}}$ . . . . .	32
<b>5</b>	<b>Resummation of the Higgs rapidity distribution</b>	<b>36</b>
5.1	Pointlike effective interaction . . . . .	36
5.2	Finite top mass . . . . .	37
5.3	Matching to the effective theory . . . . .	39
<b>6</b>	<b>Conclusions and Outlook</b>	<b>42</b>
<b>A</b>	<b>The <math>\overline{\text{MS}}</math> subtraction procedure to all orders</b>	<b>44</b>
<b>B</b>	<b>Resummation of rapidity distributions in <math>\overline{\text{MS}}</math></b>	<b>47</b>

# 1 Introduction

Techniques for the computation of small  $x$  logarithmically enhanced contributions to hard QCD cross-sections to all orders have been available for some time, and have been successfully applied to a variety of hard processes: heavy quark photo- and lepto-production [1], Deep-Inelastic Scattering [2], and more recently hadroproduction processes, including heavy quarks [3], Higgs (both in the infinite top mass limit [4] and for finite  $m_{\text{top}}$  [5, 6]), Drell-Yan [7] and prompt-photon [8]. In each of these cases, the leading contribution to the hard cross-section in the  $x \rightarrow 0$  limit has been determined to all orders in the strong coupling  $\alpha_s(Q^2)$ , where  $Q^2$  is the hard scale of the process (typically related to the invariant mass of the final state for hadroproduction processes), and  $x$  is the dimensionless ratio of this scale to the available (partonic) center-of-mass energy  $x = Q^2/s$ .

The interest in these results is twofold. First, they provide information on unknown higher-order contributions (or cross-checks on known ones): for example, the NNLO leading small  $x$  contribution to Higgs production in the finite  $m_{\text{top}}$  case [5] was used in Ref. [9] to construct an approximate expression for the yet unknown full NNLO term by combining it with an expansion in  $1/m_{\text{top}}$  [10, 11]. Second, they can be used to construct all order resummed results, which may be necessary in the kinematic region where  $\alpha_s \ln \frac{1}{x} \sim 1$  so that the conventional perturbative expansion becomes unstable. For many processes mentioned above, such as Deep-Inelastic Scattering or Drell-Yan, the Born cross-section is a quark-induced  $O(\alpha_s^0)$  electroweak process (parton model), so that the leading-log resummation of hard cross-sections is only consistent (i.e. resummed results are only factorization-scheme invariant) if combined with a next-to-leading log resummation of evolution equations. Such a consistent resummation is possible because not only leading [12] but also next-to-leading [13]  $\ln \frac{1}{x}$  contributions to evolution equations are known since some time, as well as methods to use resummed evolution equations in a way compatible with physical constraints [14, 15], and to combine solutions to resummed evolution equations with resummed hard partonic cross-sections [16, 17].

The usefulness of these results for phenomenology has been hampered so far by the fact that they, and the computational techniques on which they are based, are only available for fully inclusive cross-sections. This is likely to become a very serious limitation at the LHC where, for instance, the LHCb experiment will measure Drell-Yan rapidity distributions at very small invariant masses (i.e. at very small  $x$ ) but in a limited rapidity range [18]. It is the purpose of this paper to overcome this limitation by extending the resummation formalism to rapidity distributions.

Available calculations at small  $x$  are based on a suitable formulation of QCD factorization which is consistent with all-order leading log  $x$  resummation, called high energy or  $k_T$  factorization [1, 19]. Generalization of this factorization to the less inclusive case is a prerequisite for resummation, and it will be accomplished here. We will proceed as follows. First, in Sect. 2 we will rederive the standard high energy factorization for inclusive cross-sections of Refs. [1, 19]. This derivation is

completely equivalent to that of Refs. [1, 19], but it differs from it because the ladder expansion which underlies the factorization is treated in a way which is “dual” to it in the sense of Ref. [20]: small  $x$  logs are factored as part of standard DGLAP splitting functions, rather than by solving a BFKL [12] equation for the gluon Green function.

The consequence of this is that the same kinematics as in the proof of collinear factorization [21] can be used. Within this kinematics, the dependence on transverse and longitudinal momentum components are kept separate from each other, and this renders the generalization to rapidity distributions possible. This result is derived in Sect. 3, where a simple formula for the computation of leading log  $x$  contributions to rapidity distributions to all orders is arrived at, in terms of the Fourier-Mellin transform of the partonic rapidity distribution computed at the lowest nontrivial order with incoming off-shell gluons.

In the second part of the paper we apply this formalism to Higgs production in gluon-gluon fusion. First, in Sect. 4 we summarize the small  $x$  behavior of rapidity distributions in this case, both in the limit  $m_{\text{top}} \rightarrow \infty$  and for finite top mass, and specifically we obtain explicit results for LL $x$  terms up to NNLO, already known at NLO in the  $m_{\text{top}} \rightarrow \infty$  limit, but otherwise determined here for the first time in closed form. In Sect. 5 we then apply to this case the resummation formula of Sect. 3, by deriving a resummed expression and explicit coefficients up to NNLO. We check that the results up to  $O(\alpha_s^4)$  of the previous section are reproduced (which turns out to happen in a rather nontrivial way), we construct an explicit analytic matched  $O(\alpha_s^3)$  expression in the finite  $m_{\text{top}}$  case, and compare it to known numerical results [26, 27]. We also give an estimate of finite mass corrections at NNLO. Technical results on  $\overline{\text{MS}}$  factorization and subtraction of collinear singularities both at the inclusive and differential level are collected in two appendices.

## 2 High energy factorization of inclusive cross - sections

We will rederive high energy factorization [1, 19] in the leading logarithmic approximation (LL $x$ ) for inclusive quantities. After some introductory comments, we will first consider the known case of  $n$ -gluon emission at the double log (i.e. LL $x$ -LL $Q^2$ ) level, and finally the nontrivial general LL $x$  case. We then recall how the factorized result can be exploited for resummation, and finally we compare directly our approach to the original one of Refs. [1, 19]. The key idea of our argument is to exploit the factorization of the partonic cross-section in terms of a hard part and a ladder part, but then compute the ladder part using standard collinear factorization and DGLAP evolution, rather than BFKL evolution, exploiting the fact that due to perturbative duality [20, 28] they are in fact equivalent at the leading twist level.

## 2.1 Factorized cross-section

In order to simplify our derivation, we consider at first a photo- or lepto-production process

$$\mathcal{V}(n) + g(p) \rightarrow \mathcal{S} + X. \quad (1)$$

characterized by a hard scale  $Q^2$ . In Eq. (1)  $g(p)$  denotes a gluon of momentum  $p$ ,  $\mathcal{V}(n)$  denotes an electroweak gauge boson of momentum  $n$ , not necessarily on-shell: e.g. for neutral-current Deep-Inelastic Scattering  $\mathcal{V} = \gamma^*$  or  $\mathcal{V} = Z^*$ , and  $\mathcal{S}$  the desired final state (a quark for DIS, a dilepton for Drell-Yan and so forth). Generalization to hadroproduction or quark-initiated processes is straightforward and will be discussed In Sect. 2.6.

Our starting point is the same of Refs. [1, 19], namely the observation [12, 29] that in an axial gauge high energy enhanced terms only come from cut diagrams which are at least two-gluon reducible in the  $t$ -channel (see Fig. 1). The generic dimensionless partonic cross-section  $\sigma$  can then be split in a (process-dependent) two-gluon irreducible part (“hard” part), and a generally two-gluon reducible part (“ladder” part):

$$\sigma = \int \frac{Q^2}{2s} H^{\mu\nu}(n, p_L, p_{\mathcal{F}}, \mu_R, \mu_F, \alpha_s) \cdot L_{\mu\nu}(p_L, p, \mu_R, \mu_F, \alpha_s) [dp_L], \quad (2)$$

where the hard part  $H^{\mu\nu}$  includes the phase space for the observed final state  $\mathcal{S}$  and the momentum conservation delta function and we have also made explicit the flux factor  $1/(2s)$ . Here  $p_{\mathcal{F}}$  stands for a set of momenta parametrizing the final state of the hard part, which we will henceforth denote as  $\mathcal{F}$ , see Fig. 2. The measure  $[dp_L]$  is the  $p_L$ -loop integration measure for the cut diagram Fig. 1 and will be specified in the following (see Eq. (7) below).

Diagrams like Fig. 1 contain in general both ultraviolet and infrared divergences, which must be regulated. This leads to the dependence of the partonic cross-section on the renormalization  $\mu_R$  and factorization  $\mu_F$  scales. The renormalization scale dependence can be reabsorbed in the running coupling  $\alpha_s(\mu_R)$ . Since running coupling effects are NLL $x$ , while we are working at the LL $x$  level, in the following we will omit all the  $\mu_R$  dependence and consider only the dependence on  $\mu_F$ , which from now on we will call just  $\mu$ .

In order to simplify our derivation, we will assume further that the hard part is not just 2-gluon irreducible (2GI), but in fact 2-particle irreducible (2PI). This implies that we can neglect the  $\mu_F = \mu$  dependence in  $H^{\mu\nu}$ . The generalization to the case in which the hard part is 2GI but two-particle reducible has been worked out in [2]; the extension of our formalism to it would be straightforward but we will not discuss it further. The functions  $H^{\mu\nu}$  and  $L^{\mu\nu}$  thus depend on the following variables:

$$\begin{aligned} H^{\mu\nu} &= H^{\mu\nu}(n, p_L, p_{\mathcal{F}}, \alpha_s) \\ L^{\mu\nu} &= L^{\mu\nu}(p_L, p, \mu, \alpha_s) \end{aligned} \quad (3)$$

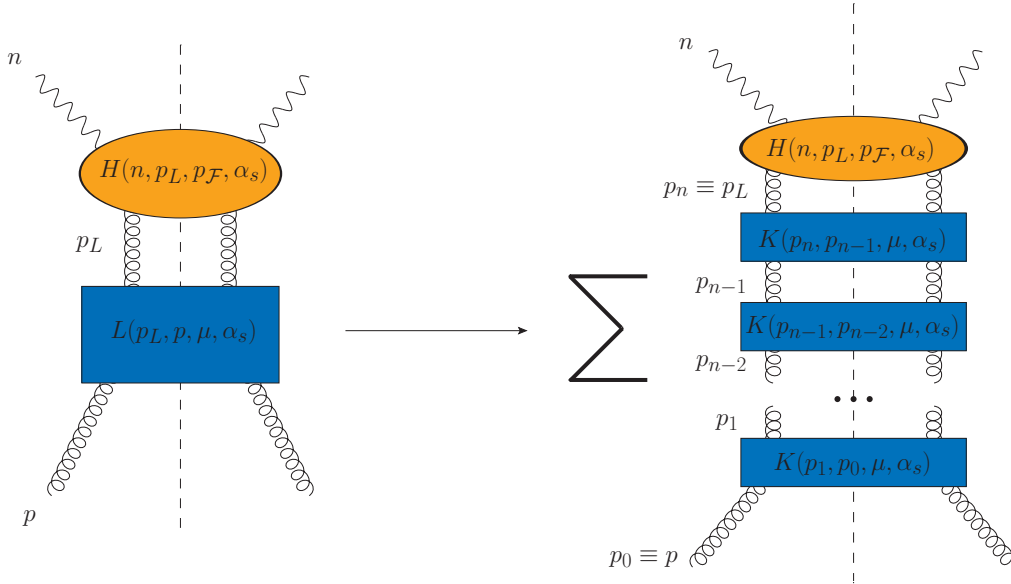


Figure 1: Left: decomposition of the cut partonic cross-section in terms of two-gluon irreducible hard part and a reducible ladder part. Right: generalized ladder expansion of the ladder part.

Since everything is on-shell and we are working in an axial gauge, we can decompose both the hard part and the ladder part in terms of conserved Lorentz structures time dimensionless scalar functions:

$$H^{\mu\nu}(n, p_L, p_F, \alpha_s) = \left(-g^{\mu\nu} + \frac{p_L^\mu p_L^\nu}{p_L^2}\right) H_\perp\left(\frac{Q^2}{n \cdot p_L}, \frac{-p_L^2}{Q^2}, \Omega_F, \alpha_s\right) + p_L^2 \left(\frac{p_L^\mu}{p_L^2} - \frac{n^\mu}{n \cdot p_L}\right) \left(\frac{p_L^\nu}{p_L^2} - \frac{n^\nu}{n \cdot p_L}\right) H_\parallel\left(\frac{Q^2}{n \cdot p_L}, \frac{-p_L^2}{Q^2}, \Omega_F, \alpha_s\right) \quad (4)$$

$$L^{\mu\nu}(p_L, p, \mu, \alpha_s) = \frac{1}{p_L^2} \left(-g^{\mu\nu} + \frac{p_L^\mu p_L^\nu}{p_L^2}\right) L_\perp\left(\frac{-p_L^2}{p \cdot p_L}, \frac{\mu^2}{-p_L^2}, \alpha_s\right) + \left(\frac{p_L^\mu}{p_L^2} - \frac{p^\mu}{p \cdot p_L}\right) \left(\frac{p_L^\nu}{p_L^2} - \frac{p^\nu}{p \cdot p_L}\right) L_\parallel\left(\frac{-p_L^2}{p \cdot p_L}, \frac{\mu^2}{-p_L^2}, \alpha_s\right) \quad (5)$$

where  $\Omega_F$  stands for a set of (typically angular, see e.g. [7]) dimensionless variables which characterize the final state  $\mathcal{F}$ . Note that we have explicitly extracted the propagator factor  $1/p_L^2$  from the scalar functions  $L_{\parallel, \perp}$ .

In the high energy limit Eq. (2) simplifies somewhat. To see this, we work in the center-of-mass frame of the colliding partons and introduce the Sudakov parametrization

$$p_L = zp - k - \frac{k_T^2}{s(1-z)}n = \left(\sqrt{\frac{s}{2}}z, -\frac{k_T^2}{\sqrt{2}s(1-z)}; -\mathbf{k}_T\right) \quad (6)$$

where  $k = (0, k_x, k_y, 0)$  is a purely transverse four-vector with  $k^2 = -k_T^2 < 0$  and as usual  $s = 2p \cdot n$ , and in the last step we have written the corresponding light-cone components. With this parametrization it is possible to show that the integration measure  $[dp_L]$  for the cut diagram Fig. 1 is (see e.g. [21])

$$[dp_L] = \frac{dz}{2(1-z)} d^2 \mathbf{k}_T. \quad (7)$$

Since the  $z$  integration goes from  $x$  to 1, we can easily identify the leading small  $x$  region with the  $z \ll 1$  region. As for  $k_T^2$ , it is typically bounded by the hard scale  $Q^2$ . Since in the high energy regime  $Q^2 \ll s$ , we have  $k_T^2/s \ll 1$ . Summarizing, at small  $x$  we are in the regime

$$z \ll 1; \quad \frac{k_T^2}{s} \ll 1; \quad (8)$$

subleading terms in  $z$  and  $k_T^2/s$  upon integration lead to power-suppressed  $O(x)$  terms.

This kinematics leads to a simplification in the arguments of the scalar functions  $H_{\parallel, \perp}$ ,  $L_{\parallel, \perp}$ . We have

$$\begin{aligned} H_{\parallel, \perp} \left( \frac{Q^2}{n \cdot p_L}, \frac{-p_L^2}{Q^2}, \Omega_{\mathcal{F}}, \alpha_s \right) &= H_{\parallel, \perp} \left( \frac{Q^2}{zs}, \frac{k_T^2}{Q^2(1-z)}, \Omega_{\mathcal{F}}, \alpha_s \right) \\ &= H_{\parallel, \perp} \left( \frac{Q^2}{zs}, \frac{k_T^2}{Q^2}, \Omega_{\mathcal{F}}, \alpha_s \right) (1 + O(z)) \\ L_{\parallel, \perp} \left( \frac{-p_L^2}{p \cdot p_L}, \frac{\mu^2}{-p_L^2}, \alpha_s \right) &= L_{\parallel, \perp} \left( -\frac{1}{2}, \frac{\mu^2}{k_T^2}, \alpha_s \right) (1 + O(z)) \\ &= L_{\parallel, \perp} \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) (1 + O(z)). \end{aligned} \quad (9)$$

With this in mind, and using the fact [1] that the scalar functions  $H_{\perp}$  and  $H_{\parallel}$  have the same small  $x$  behavior, and similarly  $L_{\perp}$  and  $L_{\parallel}$ , we can write the differential dimensionless cross-section as

$$\begin{aligned} d\sigma &= -\frac{Q^2}{2s} \left[ \frac{2}{z^2} H_{\parallel} \left( \frac{Q^2}{zs}, \frac{k_T^2}{Q^2}, \Omega_{\mathcal{F}}, \alpha_s \right) L_{\parallel} \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) (1 + O(z)) \right] dz \frac{d^2 \mathbf{k}_T}{k_T^2} = \\ &= \left[ -\frac{Q^2}{2sz} H_{\parallel} \left( \frac{Q^2}{zs}, \frac{k_T^2}{Q^2}, \Omega_{\mathcal{F}}, \alpha_s \right) (1 + O(z)) \right] \left[ 2\pi L_{\parallel} \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) \right] \frac{dz}{z} \frac{dk_T^2}{k_T^2} \frac{d\theta}{2\pi}. \end{aligned} \quad (10)$$

In Ref. [1] the ladder part  $L_{\mu\nu}(p_L, p, \mu, \alpha_s)$  is computed at the LL $x$  level in terms of a gluon Green function, which in turns sums leading logs of  $x$  by iterating a BFKL [12] kernel. Here we observe that we may equivalently express  $L_{\mu\nu}(p_L, p, \mu, \alpha_s)$  in terms of the generalized ladder expansion of Ref. [21], and we will thus express it in terms of standard collinear anomalous dimensions. First, however, we discuss the hard part.

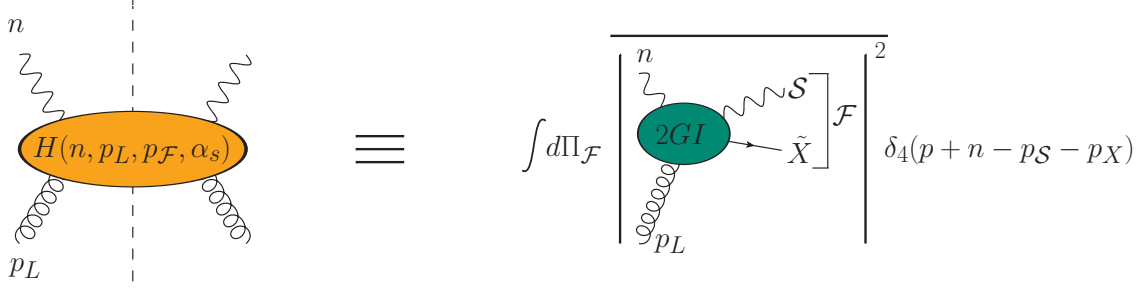


Figure 2: Graphic representation of the hard part in Eq. (2). Note that the hard part contains the momentum conservation delta function as well as the  $\mathcal{S}$  and  $\tilde{X}$  phase space integration, but it does not contain phase space integration for gluons emitted along the ladder.

## 2.2 Hard part

Let us now concentrate on the process-dependent hard part. We introduce a hard coefficient function  $C$  defined as

$$\begin{aligned} C\left(\frac{Q^2}{z_s}, \frac{Q^2}{k_T^2}, \alpha_s\right) &\equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{Q^2}{2sz} [\mathcal{P}_{\mu\nu} H^{\mu\nu}(n, p_L, p_F, \alpha_s)] = \\ &= - \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{Q^2}{2sz} H_{\parallel}\left(\frac{Q^2}{z_s}, \frac{Q^2}{k_T^2}, \Omega_F, \alpha_s\right) (1 + O(z)), \end{aligned} \quad (11)$$

where we have defined the projector

$$\mathcal{P}^{\mu\nu} \equiv \frac{k^\mu k^\nu}{k_T^2} \quad (12)$$

which, up to  $O(z)$  terms, selects the longitudinal part  $H_{\parallel}$  of the full  $H^{\mu\nu}$ . In Eq. (11) the explicit dependence of  $C$  on  $\Omega_F$  is understood.

Equation (11) has a simple interpretation:  $C$  is the cross-section for the partonic process  $\mathcal{V}(n) + g^*(q) \rightarrow \mathcal{F}$  for an off-shell incoming gluon with momentum

$$q = zp + k; \quad q^2 = -k_T^2. \quad (13)$$

In this interpretation,  $\mathcal{P}$  can be thought of as the sum over the polarizations of the off-shell gluon. Note that

$$\langle \mathcal{P}^{\mu\nu} \rangle_\theta \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \mathcal{P}^{\mu\nu} = \frac{1}{2} \left( -g_{\mu\nu} + \frac{q_\mu n_\nu + q_\nu n_\mu}{q \cdot n} \right) \equiv \frac{d_{\mu\nu}}{2}, \quad (14)$$

i.e. the azimuthal average of  $\mathcal{P}^{\mu\nu}$  performs the average over the polarizations of an on-shell gluon with momentum  $q$ . Equations (11, 14) then imply

$$\lim_{k_T^2 \rightarrow 0} \left\langle C\left(\frac{Q^2}{z_s}, \frac{Q^2}{k_T^2}, \alpha_s\right) \right\rangle_\theta = \sigma_{\text{on-shell}}(\mathcal{V}(n), g(zp) \rightarrow \mathcal{F}), \quad (15)$$



i.e. in the limit  $k_T/Q^2 \rightarrow 0$  the azimuthal average of the hard coefficient function  $C$  reduces to the on-shell partonic (dimensionless) cross-section. Note that in high energy kinematics Eq. (8), unlike in standard collinear kinematics, we cannot assume  $k_T^2 \ll Q^2$  and thus the full dependence on  $\frac{k_T^2}{Q^2}$  must be retained in the coefficient function  $C$  Eq. (11). However, the assumption that  $k_T^2 \ll s$  is sufficient [1] for the factorization of the cross-section Eq. (2) to hold.

For the subsequent discussion, it is important to observe that in the limit  $x \rightarrow 0$  the hard part, being 2GI, is finite, i.e. it does not contain logarithmically enhanced contributions. In particular, unless the vertex of the interaction for the process  $\mathcal{V}(n) + g^*(q) \rightarrow \mathcal{F}$  is pointlike, [1]

$$\lim_{x \rightarrow 0} C \left( x, \frac{Q^2}{k_T^2}, \alpha_s \right) = 0. \quad (16)$$

This is for instance the case for Deep-Inelastic Scattering and for heavy quark, Drell-Yan or Higgs production with finite  $m_{\text{top}}$ . For Higgs production in the  $m_{\text{top}} \rightarrow \infty$  limit the effective gluon-gluon-Higgs interaction is pointlike, and in such a case [4]

$$\lim_{x \rightarrow 0} C \left( x, \frac{Q^2}{k_T^2}, \alpha_s \right) = C_0 \quad (17)$$

with  $C_0$  some constant. As it is well known, and as we will see explicitly in Sect. 4 the behavior Eq. (17) leads to double logarithmic small  $x$  singularities.

## 2.3 Ladder part: generalities and double log approximation

We now turn to the computation of the ladder part. Unlike the hard part, which is 2PI, the ladder part may contain collinear singularities. In general, the collinear singularities of the partonic cross-section  $\sigma$ , Eq. (2), originate from those of the ladder part, and from the collinear region of the integration of the momentum  $p_L$  which connects the hard and ladder part. Collinear singularities can be subtracted from the partonic cross-section and factored in the parton distributions, to which the hard cross-section is connected by the lower line with momentum  $p$ , using the iterative procedure of Ref. [21], based on a generalized ladder expansion. As already mentioned, after regularization and factorization of the collinear singularities the ladder part is a function of  $\frac{k_T^2}{\mu^2}$ , where  $\mu$  is the (factorization) scale introduced in the course of regularization.

In the generalized ladder expansion,  $L_{||}$  is written as a sum of terms each of which, after regularization and factorization, leads to a contribution proportional to  $\ln^n \frac{k_T^2}{\mu^2}$  to the cross-section, with  $n$  ranging from one to infinity. The  $n$ -th order contribution comes from a ladder diagram obtained iterating  $n$  times a kernel  $K(p_i, p_{i-1}, \mu, \alpha_s)$  with  $i = 1, 2, \dots, n$ . To simplify our notation, from now on we will make the identifications  $p_n \equiv p_L$ ,  $p_0 \equiv p$ , see Fig. 1. Subsequent iterations of the kernel are connected by a pair of  $t$ -channel lines with momenta  $p_n$  (see Fig. 1) and ordered

transverse momenta  $k_{T,1}^2 \ll k_{T,2}^2 \ll \dots \ll k_{T,n}^2 = k_T^2$ . Because we are interested in the high energy limit, reducible lines are all gluon lines: quark contributions only appear at NLL $x$  and subsequent orders [1]. While we refer to Ref. [21] for a detailed discussion, the way the factorization of collinear singularities is performed in the generalized ladder expansion is summarized in Appendix A.

We will now show how to compute the ladder part using the generalized ladder expansion in the double logarithmic approximation, i.e. in the very simple case in which the result is determined simultaneously at the LL $x$  and LL $Q^2$ , and then turn in Sect. 2.4 to the more general LL $x$  case which we are interested in. At LL $x$ -LL $Q^2$ , the kernel  $K(p_i, p_{i-1}, \mu, \alpha_s)$  must be linear in  $\alpha_s$  because the highest power of  $\ln^n \frac{k_T^2}{\mu^2}$  in the cross-section must coincide with the order in  $\alpha_s$ . We define thus

$$K^1 \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) \equiv K_{||}^{\text{LL}Q^2} \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) \quad (18)$$

with  $K^1$  linear in  $\alpha_s$ .

Consider first the case in which the kernel  $K^1$  is only inserted once. The corresponding (integrated) contribution to the cross-section Eq. (10) is given by

$$\bar{\sigma}^1 \left( \frac{Q^2}{s}, \frac{\mu^2}{Q^2}, \alpha_s \right) = \int_x^1 \frac{dz}{z} \int \frac{dk_T^2}{k_T^2} C \left( \frac{Q^2}{zs}, \frac{Q^2}{k_T^2}, \alpha_s \right) \left[ 2\pi K^1 \left( \frac{\mu^2}{k_T^2}, \alpha_s \right) \right], \quad (19)$$

where the bar over  $\sigma$  indicates that collinear singularities have still to be subtracted from the cross-section, so this expression only makes sense at a regularized level. It is convenient to introduce the dimensionless variables

$$x \equiv \frac{Q^2}{s}; \quad \xi \equiv \frac{k_T^2}{Q^2}, \quad (20)$$

in terms of which Eq. (19) becomes

$$\bar{\sigma}^1 \left( x, \frac{\mu^2}{Q^2}, \alpha_s \right) = \int_x^1 \frac{dz}{z} \int \frac{d\xi}{\xi} C \left( \frac{x}{z}, \xi, \alpha_s \right) \left[ 2\pi K^1 \left( \frac{\mu^2}{Q^2 \xi}, \alpha_s \right) \right]. \quad (21)$$

Note that in terms of these variables

$$\frac{\mu^2}{Q^2 \xi} = \frac{\mu^2}{k_T^2}, \quad (22)$$

i.e. the combination  $Q^2 \xi / \mu^2$  is  $Q^2$  independent, and a  $\ln^n \frac{k_T^2}{\mu^2}$  term shows up as  $\ln^n \frac{Q^2 \xi}{\mu^2}$ .

The  $z$  convolution in Eq. (21) is turned into an ordinary product by Mellin transformation:

$$f(N) \equiv \int_0^1 dx x^{N-1} f(x); \quad f(x) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} f(N) \quad (23)$$

(note that by slight abuse of notation we denote with the same letter both the function and its transform). We get

$$\bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s \right) = \int \frac{d\xi}{\xi} C(N, \xi, \alpha_s) \left[ \frac{2\pi}{N} K^1 \left( \frac{\mu^2}{Q^2 \xi}, \alpha_s \right) \right]. \quad (24)$$

Since in the limit  $\xi \rightarrow 0$  (i.e.  $k_T^2 \rightarrow 0$ ) the coefficient function  $C$  does not vanish (see Eq. (15)), Eq. (24) is collinear divergent. We use dimensional regularization and let  $d = 4 - 2\epsilon$ . It can be shown [2] that in  $d$  dimensions  $\mathcal{P}^{\mu\nu}$  is still given by Eq. (12), hence Eq. (15) remains true also in  $d = 4 - 2\epsilon$  dimensions.

The regularized expression is explicitly given by<sup>1</sup>:

$$\bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C(N, \xi, \alpha_s; \epsilon) \left[ \frac{2\pi}{N} K^1 \left( \left( \frac{\mu^2}{Q^2} \right)^\epsilon \alpha_s; N, \epsilon \right) \right], \quad (25)$$

where the dependence of  $K^1$  on  $N$  is  $O(\epsilon)$ . In Eq. (25) the limit  $\xi \rightarrow \infty$  is never reached, because  $k_T^2$  is bounded by the kinematics (typically  $k_T^2 < Q^2$ ). Note that we have used the assumption that the hard part is 2PI, which ensures that  $C$  is free of collinear singularities [21, 30]. The generalization to processes for which  $H$  is 2GI but not 2PI (such as Deep-Inelastic Scattering and Drell-Yan) requires also treating the collinear singularities in  $H$  [2, 7]. Note that the dependence on  $\mu^2$  enters only through the combination  $(\mu^2)^\epsilon \alpha_s$  and it is thus fixed by dimensional analysis.

Using the expansion

$$\frac{1}{\xi^{1+\epsilon}} = -\frac{\delta(\xi)}{\epsilon} + \sum_{i=0}^\infty \left[ \frac{\ln^i \xi}{\xi} \right]_+ \frac{(-\epsilon)^i}{i!} \quad (26)$$

in Eq. (25) we get

$$\bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) = -\frac{1}{\epsilon} \sigma_{\text{on-shell}} \times \left[ \frac{2\pi}{N} K^1(\alpha_s) \right] + \text{finite}, \quad (27)$$

where  $\sigma_{\text{on-shell}}$  is the on-shell cross-section for the process  $\mathcal{V}(n) + g(zp) \rightarrow \mathcal{F}$ , see Eq. (15). Equation (27) shows that  $2\pi K^1(\alpha_s)/N$  is the residue of the  $O(\alpha_s)$  collinear pole, i.e. the leading-order anomalous dimension. Because all the computation has been performed at the LL $x$  level, only the leading small  $x$  contribution to the anomalous dimension is significant, so

$$\begin{aligned} \frac{2\pi}{N} K^1(\alpha_s) &= \bar{\alpha}_s \gamma_0(N); & \bar{\alpha}_s &\equiv \frac{\alpha_s N_c}{\pi} \\ \gamma_0(N) &= 1/N. \end{aligned} \quad (28)$$

Having determined  $K^1$ , we can substitute it in Eq. (25):

$$\bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) = \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C(N, \xi, \alpha_s; \epsilon) \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N), \quad (29)$$

---

<sup>1</sup>Henceforth, to simplify notation we omit terms like  $(4\pi)^\epsilon/\Gamma(1-\epsilon)$ , which in the  $\overline{\text{MS}}$  scheme are subtracted anyway. Full details on the calculation are given in Appendix A.

where we used the fact that the LO gluon-gluon anomalous dimension is  $\epsilon$  independent. Equation (29) is the expression for the regularized (collinear singular) cross-section  $\mathcal{V}(n) + g(p) \rightarrow \mathcal{F} + g$  in the small  $x$  limit.

We are now ready to compute the ladder expansion of  $L$  at  $LLx, LLQ^2$ . The regularized contribution to the cross-section when the kernel is iterated  $n$  times is

$$\begin{aligned} \bar{\sigma}^n \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) &= \left[ \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N) \right] \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C(N, \xi_n, \alpha_s; \epsilon) \times \\ &\times \int_0^{\xi_n} \left[ \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N) \right] \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \dots \times \int_0^{\xi_2} \left[ \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N) \right] \frac{d\xi_1}{\xi_1^{1+\epsilon}}. \end{aligned} \quad (30)$$

Had we used a collinear cutoff  $\mu_F$  instead of dimensional regularization, the ordered integrations in Eq. (30) would give rise to the expected  $\ln^n Q^2 / \mu_F^2$  term. In dimensional regularization, Eq. (30) contains collinear  $\epsilon$  poles. Their  $\overline{\text{MS}}$  factorization using the generalized ladder expansion of Ref. [21] is performed by requiring Eq. (30) to be finite after each  $\xi_i$  integration. This is done subtracting iteratively in such a way that Eq. (30) is finite also at the integrand level; details are given in Appendix A. This then leaves a single  $n$ -th order  $\epsilon$  pole in the cross-section, which can be subtracted using standard  $\overline{\text{MS}}$ .

The result after iterative subtraction of the first  $n - 1$  singularities is

$$\begin{aligned} \sigma^n \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) &= \left[ \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N) \right] \times \\ &\times \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C(N, \xi_n, \alpha_s; \epsilon) \frac{1}{(n-1)!} \frac{1}{\epsilon^{n-1}} \left[ \bar{\alpha}_s \gamma_0(N) \left( 1 - \left( \frac{\mu^2}{Q^2 \xi_n} \right)^\epsilon \right) \right]^{n-1} \end{aligned} \quad (31)$$

which contains a single  $1/\epsilon^n$  pole, as it ought to in the  $\overline{\text{MS}}$  scheme.

The full  $LLx$ - $LLQ^2$  result is found by adding up all contributions Eq. (31). It turns out that (as well known from Ref. [1] for the general  $LLx$  case) the sum is actually finite, i.e. the exponentiation of the leading collinear singularities to all orders leads effectively to a cross-section evaluated with off-shell gluons, which is thus finite. Indeed, we get

$$\sigma = \sum_{n=1}^{\infty} \sigma^n = \left[ \bar{\alpha}_s \left( \frac{\mu^2}{Q^2} \right)^\epsilon \gamma_0(N) \right] \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C(N, \xi, \alpha_s; \epsilon) e^{\bar{\alpha}_s \gamma_0(N) \frac{1}{\epsilon} \left[ 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon \right]}, \quad (32)$$

which is finite in the  $\epsilon \rightarrow 0$  limit. Indeed

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\xi^{1+\epsilon}} \exp \left[ \bar{\alpha}_s \gamma_0(N) \frac{1}{\epsilon} \left( 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon \right) \right] = \xi^{\bar{\alpha}_s \gamma_0(N) - 1} \exp \left[ \alpha_s \gamma_0(N) \ln \frac{Q^2}{\mu^2} \right], \quad (33)$$

and since  $\bar{\alpha}_s \gamma_0(N) > 0$  also the last collinear divergence in Eq. (32) is regulated.

We can then safely go back to four dimensions. For completeness we reintroduce the argument of the running coupling  $\alpha_s$ , although running coupling is a  $NLLx$

effect. We obtain

$$\sigma \left( N, \frac{\mu^2}{Q^2}, \alpha_s(\mu^2) \right) = \bar{\alpha}_s(\mu^2) \gamma_0(N) \times \int_0^\infty d\xi \xi^{\bar{\alpha}_s(\mu^2) \gamma_0(N) - 1} e^{\bar{\alpha}_s(\mu^2) \gamma_0(N) \ln \frac{Q^2}{\mu^2}} C(N, \xi, \alpha_s(\mu^2)), \quad (34)$$

or, using renormalization group invariance to set  $\mu^2 = Q^2$ , thereby absorbing the explicit  $\mu^2/Q^2$  dependence into the running coupling  $\alpha_s(Q^2)$ ,

$$\sigma(N, \alpha_s(Q^2)) = \bar{\alpha}_s(Q^2) \gamma_0(N) \int_0^\infty d\xi \xi^{\bar{\alpha}_s(Q^2) \gamma_0(N) - 1} C(N, \xi, \alpha_s(Q^2)), \quad (35)$$

where we have made explicit the fact that the partonic cross-section no longer depends on  $\mu^2/Q^2$ . At each fixed order in  $\alpha_s$  Eq. (35) reproduces the small  $N$  limit of the coefficient function, as can be checked by integrating by parts and expanding in powers of  $\alpha_s$ .

We see that there are two sources for the  $N$ -dependence of the cross-section  $\sigma$ : the  $\bar{\alpha}_s \gamma_0(N)$  terms and the explicit  $N$ -dependence of  $C$ . Let us first assume that  $C(N, \xi, \alpha_s)$  is regular in  $N = 0$  (i.e.  $C(x, \xi, \alpha_s) \rightarrow 0$  for  $x \rightarrow 0$ , see the discussion at the end of Sec. 2.2):

$$C(N, \xi, \alpha_s) = C(0, \xi, \alpha_s) + NC'(0, \xi, \alpha_s) + \dots \quad (36)$$

In this case Eq. (35) can be written as (see Eq. (28))

$$\begin{aligned} \sigma(N, \alpha_s) &= \frac{\bar{\alpha}_s}{N} \int d\xi \xi^{\frac{\bar{\alpha}_s}{N} - 1} C(0, \xi, \alpha_s) (1 + O(N)) \\ &= \alpha_s^k \sum_{i=0}^{\infty} c_i^{\text{LL}x - \text{LL}Q^2} \left( \frac{\bar{\alpha}_s}{N} \right)^i (1 + O(N)), \end{aligned} \quad (37)$$

where the power  $k$  is process-dependent and is encoded in  $C$  (e.g.  $k = 1$  for Drell-Yan and DIS,  $k = 2$  for Higgs). The expansion in  $\alpha_s/N$  leads to the desired  $\alpha_s \ln \frac{1}{x}$  expansion in  $x$  space. From Eq. (37) it is then clear that only the  $N$ -independent term  $C(0, \xi, \alpha_s)$  is relevant for a  $\text{LL}x$  calculation. Things are different when  $C(x, \xi, \alpha_s) \rightarrow C_0 \neq 0$  for  $x \rightarrow 0$ . This is for instance the case for the Higgs coefficient function in the infinite  $m_{\text{top}}$  approximation [4]. In this case  $C(N, \xi, \alpha_s)$  is not regular in  $N = 0$ , and the full  $N$ -dependence must be kept. The singular behavior of the coefficient function in  $N = 0$  leads to double log behavior in  $x$  space, as we shall see explicitly in Sec. 5.

Eq. (35) reproduces the  $\text{LL}x$  result of Ref. [1] in the  $\text{LL}Q^2$  case: the  $\text{LL}x$  coefficient function is found evaluating the partonic cross-section with incoming off-shell gluon, performing a Mellin transform with respect to the gluon virtuality, and then letting  $M = \gamma$ , where  $M$  is the Mellin variable which is conjugate to the gluon virtuality, and  $\gamma$  is the relevant anomalous dimension, which in the  $\text{LL}Q^2$  case coincides with the small  $x$  limit of the leading order anomalous dimension  $\gamma_0$ .

## 2.4 Ladder part: LLx case

The argument presented in Sect. 2.4 actually also works in the general LLx case, at the price of some technical complication. In this case the kernel  $K$  contains contributions to all orders in the strong coupling  $\alpha_s$ . The way the ladder expansion is constructed when the kernel  $K$  is defined at a generic order in  $\alpha_s$  was explained in Ref. [21], and it is that which is used to compute anomalous dimensions at any perturbative order. In a nutshell, the ladder diagram of Fig. 1 is still constructed with strongly ordered transverse momenta in such a way that all large logs of  $Q^2$  arise due to integration over the momenta  $p_i$ . Unordered contributions, as well as finite parts due to iterations of lower-order kernel after subtraction of collinear poles, are included as higher order contributions to the kernel  $K$ , which is now no longer 2PI, even though it is still free of collinear singularities. The total cross-section is still the sum of contributions with  $n$  iterations of the kernel, with  $n$  ranging from one to infinity.

With a generic kernel, the steps leading to Eq. (25) remain unchanged, and lead to

$$\begin{aligned} \bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) &= \\ &= \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C(N, \xi, \alpha_s; \epsilon) K \left( N, \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon, \alpha_s; \epsilon \right) \end{aligned} \quad (38)$$

Because  $K$  is free of collinear singularities, the analogue of Eq. (27) also still holds:

$$\bar{\sigma}^1 \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) = -\frac{1}{\epsilon} \sigma_{\text{on-shell}} K(N, 1, \alpha_s) + \text{finite}, \quad (39)$$

which again implies that  $K$  is just the standard anomalous dimension:

$$K \left( N, \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon, \alpha_s; \epsilon \right) = \gamma \left( N, \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon, \alpha_s; \epsilon \right). \quad (40)$$

At the LLx level at which we are working  $\gamma$  is the ( $d$ -dimensional) LLx anomalous dimension.

However, unlike in the LLQ<sup>2</sup> case, the anomalous dimension  $\gamma$  now explicitly depends on  $\epsilon$ ; this affects the way the  $\overline{\text{MS}}$  subtraction is performed. Indeed Eq. (30) now becomes

$$\begin{aligned} \bar{\sigma}^n \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) &= \int_0^\infty \left[ \gamma \left( N, \left( \frac{\mu^2}{Q^2 \xi_n} \right)^\epsilon, \alpha_s; \epsilon \right) \right] \frac{d\xi_n}{\xi_n^{1+\epsilon}} C(N, \xi_n, \alpha_s; \epsilon) \times \\ &\times \int_0^{\xi_n} \left[ \gamma \left( N, \left( \frac{\mu^2}{Q^2 \xi_{n-1}} \right)^\epsilon, \alpha_s; \epsilon \right) \right] \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \dots \times \int_0^{\xi_2} \left[ \gamma \left( N, \left( \frac{\mu^2}{Q^2 \xi_1} \right)^\epsilon, \alpha_s; \epsilon \right) \right] \frac{d\xi_1}{\xi_1^{1+\epsilon}}, \end{aligned} \quad (41)$$

which, after iterative subtraction of the first  $n - 1$  singularities becomes

$$\begin{aligned} \sigma^n \left( N, \frac{\mu^2}{Q^2}, \alpha_s; \epsilon \right) &= \gamma \left( N, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C(N, \xi_n, \alpha_s; \epsilon) \times \\ &\times \frac{1}{(n-1)!} \frac{1}{\epsilon^{n-1}} \left[ \sum_i \frac{\tilde{\gamma}_i(N, \alpha_s; 0)}{i} \left( 1 - \left( \frac{\mu^2}{Q^2 \xi_n} \right)^{i\epsilon} \frac{\tilde{\gamma}_i(N, \alpha_s; \epsilon)}{\tilde{\gamma}_i(N, \alpha_s; 0)} \right) \right]^{n-1}, \end{aligned} \quad (42)$$

where we have introduced the expansion

$$\gamma \left( N, \left( \frac{\mu^2}{Q^2 \xi} \right)^\epsilon, \alpha_s; \epsilon \right) = \sum_{i=0}^\infty \tilde{\gamma}_i(N, \alpha_s; \epsilon) \left( \frac{\mu^2}{Q^2 \xi} \right)^{i\epsilon}. \quad (43)$$

If  $\gamma$  is given at the  $k$ -th-perturbative order the sum goes from 0 to  $k$ .

Again, the total cross-section is rendered finite by exponentiation of the collinear poles:

$$\begin{aligned} \sigma &= \sum_{n=1}^\infty \sigma^n = \gamma \left( N, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C(N, \xi, \alpha_s; \epsilon) \times \\ &\times \exp \left[ \frac{1}{\epsilon} \sum_i \frac{\tilde{\gamma}_i(N, \alpha_s; 0)}{i} \left( 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^{i\epsilon} \frac{\tilde{\gamma}_i(N, \alpha_s; \epsilon)}{\tilde{\gamma}_i(N, \alpha_s; 0)} \right) \right]. \end{aligned} \quad (44)$$

However, we must keep track of the  $\epsilon$  dependence in  $\gamma_i$ . Expanding

$$\tilde{\gamma}_i(N, \alpha_s; \epsilon) \equiv \tilde{\gamma}_i(N, \alpha_s) + \epsilon \dot{\tilde{\gamma}}_i(N, \alpha_s) + \epsilon^2 \ddot{\tilde{\gamma}}_i(N, \alpha_s) + \dots \quad (45)$$

we get

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\xi^{1+\epsilon}} \exp \left[ \frac{1}{\epsilon} \sum_i \frac{\tilde{\gamma}_i(N, \alpha_s; 0)}{i} \left( 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^{i\epsilon} \frac{\tilde{\gamma}_i(N, \alpha_s; \epsilon)}{\tilde{\gamma}_i(N, \alpha_s; 0)} \right) \right] &= \\ &= \xi^{\gamma(N, \alpha_s)-1} \exp \left[ \gamma(N, \alpha_s) \ln \frac{Q^2}{\mu^2} \right] \mathcal{R}(N, \alpha_s), \\ \mathcal{R}(N, \alpha_s) &\equiv \exp \left[ - \sum_i \frac{\dot{\tilde{\gamma}}_i(N, \alpha_s)}{i} \right]. \end{aligned} \quad (46)$$

In comparison to the LL $Q^2$  result Eq. (32) the process independent but scheme dependent  $\mathcal{R}$  factor has now appeared. Note that, because we are working in the LL $x$  approximation,  $\alpha_s$  does not run. At NLL $x$  and beyond the running of the coupling generates further contributions to  $\mathcal{R}$ , that depend on higher order terms in the expansion (45) in powers of  $\epsilon$  [31, 32]. The LL $x$  cross-section is finally

$$\sigma(N, \alpha_s(Q^2)) = \gamma \left( \frac{\alpha_s(Q^2)}{N} \right) \int_0^\infty d\xi \xi^{\gamma \left( \frac{\alpha_s(Q^2)}{N} \right) - 1} C(N, \xi, \alpha_s(Q^2)) \mathcal{R} \left( \frac{\alpha_s(Q^2)}{N} \right), \quad (47)$$

where without loss of generality we have set  $\mu^2 = Q^2$  (thus omitting the explicit dependence on  $\mu^2/Q^2$ ) and we have noticed that the LLx anomalous dimension which enters Eq. (47) only depends on  $\alpha_s$  and  $N$  through the combination  $\frac{\alpha_s}{N}$ . As in the LL $Q^2$  case, if  $C(N, \xi, \alpha_s(Q^2))$  is regular in  $N = 0$ , then only  $C(0, \xi, \alpha_s(Q^2))$  is relevant at LLx (see discussion around Eq. (36)).

However, there is a further source of scheme dependence of our result. Indeed, we have seen that after exponentiation the last collinear singularity disappears; it is however present if the result is expanded out and determined order by order in perturbation theory. It is thus not obvious that the  $\overline{\text{MS}}$  subtraction commutes with exponentiation, and therefore the result Eq. (46) might be defined in a scheme which differs from  $\overline{\text{MS}}$  by a further LLx scheme change, which amounts to a redefinition of parton distributions by a factor  $\mathcal{N}(\alpha_s/N)$  [22, 23, 24].

Hence, our result has the general form

$$\sigma(N, \alpha_s(Q^2)) = \gamma\left(\frac{\alpha_s(Q^2)}{N}\right) \int_0^\infty d\xi \xi^{\gamma\left(\frac{\alpha_s(Q^2)}{N}\right)-1} C(N, \xi, \alpha_s(Q^2)) R\left(\frac{\alpha_s(Q^2)}{N}\right), \quad (48)$$

with

$$R\left(\frac{\alpha_s}{N}\right) \equiv \mathcal{N}\left(\frac{\alpha_s}{N}\right) \mathcal{R}\left(\frac{\alpha_s}{N}\right). \quad (49)$$

The factor  $\mathcal{N}$  (and thus  $R$ ) which fixes the factorization scheme has been determined for  $\overline{\text{MS}}$  through a computation of the LLx gluon Green function [19].

## 2.5 Cross-section resummation

The LLx cross-section Eq. (48) can be written in terms of the  $M$ -Mellin transform of the coefficient function  $C$

$$h(N, M, \alpha_s) \equiv M \int_0^\infty d\xi \xi^{M-1} C(N, \xi, \alpha_s) R_{\overline{\text{MS}}}(M), \quad (50)$$

supplemented by the condition

$$M = \gamma\left(\frac{\alpha_s}{N}\right), \quad (51)$$

namely

$$\sigma(N, \alpha_s) = h\left(N, \gamma\left(\frac{\alpha_s}{N}\right), \alpha_s\right). \quad (52)$$

Explicit resummed expressions can be obtained from Eq. (52) by substituting in it the well-known [12, 28] expression for the LLx anomalous dimension

$$\gamma\left(\frac{\alpha_s}{N}\right) = \frac{\bar{\alpha}_s}{N} + 2\zeta(3)\left(\frac{\bar{\alpha}_s}{N}\right)^4 + 2\zeta(5)\left(\frac{\bar{\alpha}_s}{N}\right)^6 + 12\zeta(3)^2\left(\frac{\bar{\alpha}_s}{N}\right)^7 + \dots \quad (53)$$

and for  $R$  the appropriate to the factorization scheme.



The result (50-51) coincides with that of Ref. [2], where it was derived by solving the small  $x$  evolution equation for the gluon Green function. Indeed, our derivation parallels that of Ref. [2], the difference being that we have determined the ladder part using the generalized ladder expansion of Ref. [21], while in Ref. [2] it was computed as the gluon Green function. These coincide because of DGLAP-BFKL duality [20].

Explicitly, our Eq. (10) in Ref. [2] is written in the form

$$\sigma = \int \frac{dz}{z} \frac{dk_T^2}{k_T^2} C\left(\frac{x}{z}, \frac{k_T^2}{Q^2}, \alpha_s\right) \mathcal{G}(z, k_T^2), \quad (54)$$

where  $\mathcal{G}$  is a gluon Green function, which satisfies a LL $x$  (BFKL) evolution equation and  $C$  is defined as in Eq. (11). Taking an  $M$ -Mellin transform Eq. (50) both of the coefficient function  $C$  and the Green function  $\mathcal{G}$ , and noting that in  $M$  space the Green function has the form

$$\mathcal{G}(N, M) = \frac{r(M)}{M - \gamma(\alpha_s/N)} [1 + O(M - \gamma(\alpha_s/N))], \quad (55)$$

one ends up with the expression

$$\sigma(N, \alpha_s) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} \left(\frac{\mu^2}{Q^2}\right)^{-M} h(N, M) \mathcal{G}(N, M) \quad (56)$$

of the resummed cross-section, which is immediately seen to coincide with Eq. (48) (with  $\mu^2 = Q^2$ ), by performing the  $M$  integral and identifying the  $R$  function with

$$R(\alpha_s/N) = r(\gamma(\alpha_s/N)). \quad (57)$$

The form (54) of the resummed result has the structure of a high energy factorization theorem. It is important however to note that  $\sigma$  on the left-hand side of Eq. (54) is a partonic cross-section. Namely, the basic insight of Ref. [1] is that, after factoring the hadronic cross-section into parton distributions convoluted with a partonic cross-section, at the LL $x$  level the partonic cross-section can in turn be itself factorized into a hard part and a ladder part. The argument presented in this section has pushed one step further the analogy between the factorization of the hadronic process and the subsequent factorization of the partonic cross-section, by using the standard collinear factorization technique of Ref. [21], albeit at the LL $x$  level, to compute the ladder part of this second factorization.

## 2.6 Hadroproduction and quark-initiated processes

We now discuss briefly how our results can be generalized to the case of hadroproduction processes. This differs from the case discussed so far because there are now two incoming partons. Consider the process  $g(n) + g(p) \rightarrow \mathcal{S} + X$ . We have now

two different transverse momenta  $k_T$ , one for each leg. In principle, the interference between the two legs could spoil the factorized result Eq. (2); however, these contributions do not appear at LLx [1], so the result in this case is simply found by applying the generalized ladder expansion to both legs. In particular, Eq. (2) generalizes to

$$\sigma \equiv \int \frac{Q^2}{2s} H^{\mu\nu\bar{\mu}\bar{\nu}}(n_L, p_L, p_{\mathcal{F}}, \alpha_s) \cdot L_{\mu\nu}(p_L, p, \mu, \alpha_s) L_{\bar{\mu}\bar{\nu}}(n_L, n, \mu, \alpha_s) [dp_L] [dn_L], \quad (58)$$

where again we have assumed that  $H$  is 2PI,  $\mu$  is the factorization scale and we have omitted any dependence on the renormalization scale. We have assigned bar index to contributions from the upper leg. In Eq. (58)  $n_L$  is defined through the Sudakov parametrization (see Eq. (6) for comparison)

$$n_L = \bar{z}n - \bar{k} - \frac{\bar{k}_T^2}{s(1-\bar{z})}p, \quad (59)$$

with  $\bar{k} = (0, \bar{k}_x, \bar{k}_y)$  and  $\bar{k}^2 = -\bar{k}_T^2 < 0$ . In this parametrization we have (see Eq. (7))

$$[dn_L] = \frac{d\bar{z}}{2(1-\bar{z})} d^2\bar{\mathbf{k}}_T^2 = \frac{d\bar{z}}{4(1-\bar{z})} d\bar{k}_T^2 d\bar{\theta} \quad (60)$$

In the high energy regime we have  $\bar{z} \ll 1$ ,  $\frac{\bar{k}_T^2}{s} \ll 1$  (see (8)). Given the factorized form Eq. (58), we can define a hadroproduction coefficient function  $C$ :

$$C\left(\frac{x}{z\bar{z}}, \xi, \bar{\xi}, \alpha_s\right) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{d\bar{\theta}}{2\pi} \frac{Q^2}{2sz\bar{z}} \mathcal{P}_{\mu\nu} \bar{\mathcal{P}}_{\bar{\mu}\bar{\nu}} H^{\mu\nu\bar{\mu}\bar{\nu}}(n_L, p_L, p_{\mathcal{F}}, \alpha_s), \quad (61)$$

where again we have omitted the  $\Omega_{\mathcal{F}}$  dependence and

$$\bar{\xi} \equiv \frac{\bar{k}_T^2}{Q^2}; \quad \bar{\mathcal{P}}^{\bar{\mu}\bar{\nu}} \equiv \frac{\bar{k}^{\bar{\mu}} \bar{k}^{\bar{\nu}}}{\bar{k}_T^2}. \quad (62)$$

Eq. (61) should be compared to its photo- or lepto-production counterpart Eq. (11). In particular, we now have (see Eq. (15))

$$\lim_{\xi, \bar{\xi} \rightarrow 0} \left\langle C\left(\frac{x}{z\bar{z}}, \xi, \bar{\xi}, \alpha_s\right) \right\rangle_{\theta, \bar{\theta}} = \sigma_{\text{on-shell}}(g(zn), g(zp) \rightarrow \mathcal{F}). \quad (63)$$

Starting from the hard coefficient function Eq. (61), we can repeat the same steps of the previous subsections, and arrive at the hadroproduction formula (with  $\mu^2 = Q^2$ )

$$\begin{aligned} \sigma(N, \alpha_s(Q^2)) &= \int_0^\infty d\xi R\left(\frac{\alpha_s(Q^2)}{N}\right) \gamma\left(\frac{\alpha_s(Q^2)}{N}\right) \xi^{\gamma\left(\frac{\alpha_s(Q^2)}{N}\right)-1} \times \\ &\times \int_0^\infty d\bar{\xi} R\left(\frac{\alpha_s(Q^2)}{N}\right) \gamma\left(\frac{\alpha_s(Q^2)}{N}\right) \bar{\xi}^{\gamma\left(\frac{\alpha_s(Q^2)}{N}\right)-1} C(N, \xi, \bar{\xi}, \alpha_s(Q^2)), \end{aligned} \quad (64)$$

to be compared to its photo- or lepto-production counterpart Eq. (48).

Before concluding this section, we now briefly account for quark-initiated processes. The analysis in [12] shows that at the LL $x$  level only gluons enter in the generalized ladder, with one exception: since the  $P_{gq}^0$  splitting function is singular at small  $x$ , the first rung of the ladder can be a quark. In other words, the initial quark can emit a collinear quark, thus becoming a gluon which starts radiating the generalized ladder Fig. 1, leading to a cross-section proportional to  $P_{gq}^0$ . Since at small  $x$  we have  $P_{gq}^0 = \frac{C_F}{C_A} P_{gg}^0$ , the only difference between quark- and gluon-initiated processes is a color-charge factor.

### 3 High energy factorization of rapidity distributions

The formulation of high energy factorization of Sect. 2 will now be extended to rapidity distributions, thereby leading to a method for the determination of LL $x$  contributions to all orders. The argument will closely follow that of Sect. 2.

#### 3.1 Kinematics and factorization

We now concentrate from the onset on hadroproduction of a final state system  $\mathcal{S}$  with invariant mass  $Q^2$ :

$$h_1(P_1) + h_2(P_2) \rightarrow \mathcal{S}(p_S) + X \quad (65)$$

and we study distribution in the rapidity of  $\mathcal{S}$

$$Y \equiv \frac{1}{2} \ln \frac{E_S + p_{Sz}}{E_S - p_{Sz}}, \quad (66)$$

where  $E_S, p_{Sz}$  are measured in the hadronic center-of-mass frame. The (dimensionless) hadronic rapidity distribution is written in terms of parton distributions  $f_i^h$  and a partonic rapidity distribution  $\frac{d\sigma_{ab}}{dy}$ :

$$\begin{aligned} \frac{d\sigma_h}{dY}(x_h, Y, Q^2, \alpha_s) &= \sum_{a,b} \int_{\sqrt{x_h} e^Y}^1 dx_1 \int_{\sqrt{x_h} e^{-Y}}^1 dx_2 \times \\ &\times \frac{d\sigma_{ab}}{dy} \left( \frac{x_h}{x_1 x_2}, \frac{\mu^2}{Q^2}, Y - \frac{1}{2} \ln \frac{x_1}{x_2}, \alpha_s \right) \times f_a^{h_1}(x_1, \mu^2) f_b^{h_2}(x_2, \mu^2). \end{aligned} \quad (67)$$

where  $x_h = \frac{Q^2}{S}$  and  $S = (P_1 + P_2)^2$ . Note that we have used the well-known fact that in collinear factorization the hadronic and partonic center-of-mass frames are related by a longitudinal boost.

Henceforth, as in the previous section, we work in the partonic center-of-mass frame, where  $x_1 = x_2 = 1$  (thus the partonic rapidity  $y$  is equal to the hadronic one  $Y$ ) and the partonic rapidity distribution is given by

$$\frac{d\sigma}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s \right) \equiv \int \frac{|\overline{M}|^2}{2s} d\Pi_f \delta \left( y - \frac{1}{2} \ln \frac{E_{\mathcal{S}} + p_{\mathcal{S}z}}{E_{\mathcal{S}} - p_{\mathcal{S}z}} \right) \quad (68)$$

where the bar stands for sum (average) over final (initial) helicities and colors and  $d\Pi_f$  is the final particle phase space, and here and henceforth for simplicity the parton indices  $a, b$  have been omitted.

Resummation at the differential level in rapidity can now be performed by repeating the argument of Sect. 2, but with the partonic cross-section now multiplied by the kinematic constraint enforced by the rapidity delta in Eq. (68). In the factorized expression Eq. (58) the dependence on the momentum  $p_{\mathcal{S}}$  of the final state system  $\mathcal{S}$  (see Fig. 2) is entirely contained in the hard part. Therefore, the kinematic constraint only enters the definition of the coefficient function  $C$  which now becomes

$$C_y \left( x, z, \bar{z}, \xi, \bar{\xi}, y, \alpha_s \right) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi} \frac{Q^2}{2s z \bar{z}} \times \\ \times \left[ \mathcal{P}_{\mu\nu} \bar{\mathcal{P}}_{\bar{\mu}\bar{\nu}} H^{\mu\nu\bar{\mu}\bar{\nu}}(n_L, p_L, p_{\mathcal{F}}, \alpha_s) \right] \delta \left( y - \frac{1}{2} \ln \frac{E_{\mathcal{S}} + p_{\mathcal{S}z}}{E_{\mathcal{S}} - p_{\mathcal{S}z}} \right), \quad (69)$$

where  $z, \bar{z}, \xi, \bar{\xi}$  were respectively defined in Eqs. (6, 59, 20, 62) and we have made explicit the dependence on the rapidity  $y$ , which in turn is fixed in terms of the remaining kinematic variables by the delta in Eq. (68). Note that, as in the inclusive case Eq. (61), in general the coefficient function  $C_y$  also depends on a residual set  $\Omega_{\mathcal{F}}$  of final-state variables which survive the phase space integration. This dependence is omitted for simplicity.

Equations (61, 69) show that the ladder part is unaffected by the rapidity constraint. The rapidity-dependent coefficient function  $C_y$  instead acquires a nontrivial dependence on all kinematic variables because of this constraint. However, we will now show that invariance under longitudinal boosts of the ladder implies that the dependence of  $C_y$  on the longitudinal momentum fractions  $z, \bar{z}$  is completely determined by its dependence on the rapidity  $y$  and the scaling variable  $x$ . But the argument presented in Sect. 2 shows that the effect of the ladder insertion on the external legs (which generates the  $LLx$  series) can be cast in terms of QCD evolution in  $k_T^2$ , characterized by standard anomalous dimensions. Therefore, this implies that we will be able to trade the rescaling of the momenta  $p_L$  and  $n_L$  which enter the hard part for a shift in rapidity. This will enable us to determine fully the  $LLx$  resummation of the rapidity distribution from the knowledge of the rapidity dependence of the hard part  $C_y$ .

In the remainder of this section, we will prove these results, which are essentially of kinematic origin, while the subtraction of collinear singularities in the ladder part is unchanged and can be handled as in Sect. 2. Therefore, for clarity we will simply

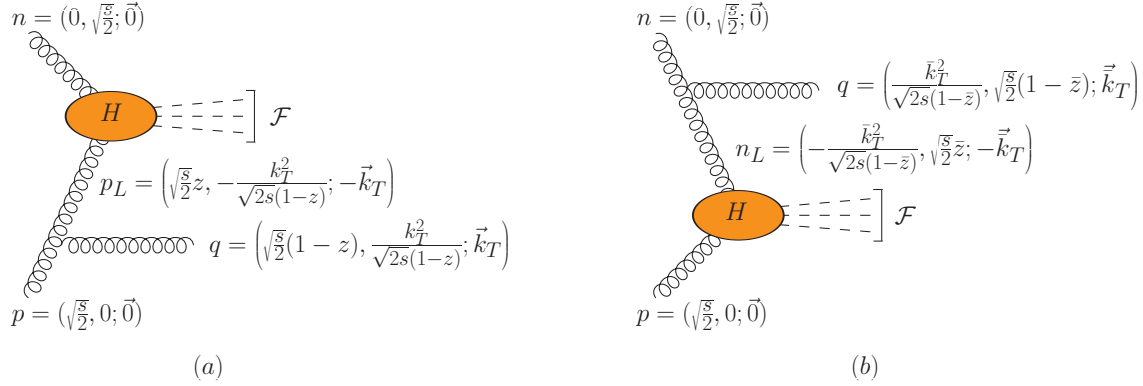


Figure 3: Single emission kinematics from both legs in terms of light-cone Sudakov components, see Eq. (6).

regulate collinear singularities with a cutoff. The price to pay for this is that we will lose full control of the factorization scheme. The full computation in the  $\overline{\text{MS}}$  scheme is presented in Appendix B, and will provide us with a determination of the scheme-dependent factor  $R$ .

### 3.2 Ladder part and rapidity dependence

As we have seen in Sect. 2, large logs of  $\frac{1}{x}$  come from gluon ladders attached to the external legs of the hard part. We now show how to compute the effect of these ladders on rapidity distributions. The structure of the computation is unchanged: the ladder is constructed by iterating  $n$  times a kernel  $K$ , which is closely related to the standard QCD anomalous dimension. In the case of single insertion of the kernel on the lower leg (say) we have

$$\begin{aligned} \frac{d\sigma^1}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s \right) &= \\ &= \int_x^1 \frac{dz}{z} \int_{\mu^2}^{\infty} \frac{dk_T^2}{k_T^2} C_y \left( \frac{x}{z}, \frac{k_T^2}{Q^2}, y \left( x, z, \frac{k_T^2}{Q^2} \right), \alpha_s \right) K(z, \alpha_s), \end{aligned} \quad (70)$$

which only differs from Eq. (19) of the inclusive case because of the replacement of the inclusive coefficient function  $C$  with the differential one  $C_y$ , and because of the kinematic constraint imposed by the delta function Eq. (68) which relates the rapidity  $y$  to the other kinematic variables of the hard part.

The kinematics of the single insertion is displayed in Fig. 3, using the notation of Eq. (6), in the LLQ<sup>2</sup> case in which the kernel corresponds to single-gluon emission as discussed in Sect. 2.3. As discussed in Sect. 2.4 the kinematics in the general LL $x$  case is the same. It is apparent that

$$p_L = \left( \sqrt{\frac{s}{2}} z, 0; -\mathbf{k}_T \right) \left( 1 + O \left( \frac{k_T^2}{s} \right) \right). \quad (71)$$

It follows that the longitudinal components of  $p_L$  are related to those of the Sudakov base momentum  $p$  by a longitudinal boost. Therefore

$$C_y(x, z, k_T^2, Q^2, y \left( x, z, \frac{k_T^2}{Q^2} \right), \alpha_s) = C_y \left( \frac{x}{z}, \frac{k_T^2}{Q^2}, y \left( x, \frac{k_T^2}{Q^2} \right) - \frac{1}{2} \ln z, \alpha_s \right) + O \left( \frac{k_T^2}{s} \right), \quad (72)$$

i.e., the effect of a  $K$ -kernel insertion on the rapidity dependence of the hard part is a longitudinal boost, up to subleading terms in small  $x$  kinematics Eq. (8).

Eq. (70) can be easily generalized to the case of  $n$ -kernel insertion, with the result:

$$\begin{aligned} \frac{d\sigma^n}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s \right) &= \int_{\mu^2}^{\infty} \frac{dk_T^2}{k_T^2} \frac{1}{(n-1)!} \ln^{n-1} \frac{k_T^2}{\mu^2} \times \\ &\times \int_x^1 \frac{dz_n}{z_n} P(z_n, \alpha_s) \times \dots \times \int_{\frac{x}{z_2 \dots z_n}}^1 \frac{dz_1}{z_1} P(z_1, \alpha_s) \times \\ &\times C_y \left( \frac{x}{z_1 \dots z_n}, \frac{k_T^2}{Q^2}, y - \frac{1}{2} \ln z_1 - \dots - \frac{1}{2} \ln z_n, \alpha_s \right), \end{aligned} \quad (73)$$

where  $P$  is the inverse Mellin transform of the kernel  $K$  and for simplicity we have written  $y(x, k_T^2/Q^2) = y$ .

Equation (73) can be factorized by taking a Fourier-Mellin transform, defined (together with its inverse) as

$$\begin{aligned} f(N, b) &= \int_0^1 dx x^{N-1} \int_{-\infty}^{\infty} dy e^{iby} f(x, y); \\ f(x, y) &= \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i} x^{-N} \int_{-\infty}^{\infty} \frac{db}{2\pi} e^{-iby} f(N, b), \end{aligned} \quad (74)$$

where as usual by slight abuse of notation we use the same symbol to denote the function and its transform. We get

$$\begin{aligned} \frac{d\sigma^n}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s \right) &= \int_{-\infty}^{\infty} dy e^{iby} \int_0^1 dx x^{N-1} \frac{d\sigma^n}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s \right) = \\ &= \int_{-\infty}^{\infty} dy e^{iby} \int_0^1 dx x^{N-1} \times \int_x^1 \frac{dz_n}{z_n} e^{i\frac{b}{2} \ln z_n} P(z_n, \alpha_s) \times \dots \times \\ &\times \int_{\frac{x}{z_2 \dots z_n}}^1 \frac{dz_1}{z_1} e^{i\frac{b}{2} \ln z_1} P(z_1, \alpha_s) \times \int_{\mu^2}^{\infty} \frac{dk_T^2}{k_T^2} \frac{\ln^{n-1} \frac{k_T^2}{\mu^2}}{(n-1)!} \times \\ &\times C_y \left( \frac{x}{z_1 \dots z_n}, \frac{k_T^2}{Q^2}, y, \alpha_s \right). \end{aligned} \quad (75)$$

Each  $z$  integral in Eq. (75) is a convolution, hence in Fourier-Mellin space the result

factorizes:

$$\frac{d\sigma^n}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s \right) = \left[ \gamma \left( N + \frac{ib}{2}, \alpha_s \right) \right]^n \int_{\mu^2}^{\infty} \frac{dk_T^2}{k_T^2} C_y \left( N, \frac{k_T^2}{Q^2}, b, \alpha_s \right) \frac{\ln^{n-1} \frac{k_T^2}{\mu^2}}{(n-1)!}. \quad (76)$$

As in Sect. 2, the final result for the cross-section is obtained by adding up the contributions from  $n$  insertions of the kernel, with  $1 \leq n \leq \infty$ . We get

$$\begin{aligned} \frac{d\sigma}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s \right) &= \sum_{n=1}^{\infty} \frac{d\sigma^n}{dy} = \\ &= \gamma \left( \frac{\alpha_s}{N + \frac{ib}{2}} \right) \int_{\mu^2}^{\infty} dk_T^2 (k_T^2)^{\gamma(\frac{\alpha_s}{N+ib/2})-1} C_y \left( N, \frac{k_T^2}{Q^2}, b, \alpha_s \right). \end{aligned} \quad (77)$$

### 3.3 Resummation of rapidity distributions

Equation (77) provides the resummation of LL $x$  terms due to radiation from one of the two outer legs of the hard kernel, so a full resummation can be obtained from its two-leg generalization. Before discussing this, we observe that in Eq. (77) the multiple poles in  $N = 0$  which in the inclusive result Eq. (46) lead to the LL $x$  terms have now become multiple poles in  $N = -ib/2$ . On top of these, the hard coefficient function  $C_y$  may still contain poles in  $N = 0$ , but, as discussed in Sect. 2.2, this only happens in the somewhat pathological case of pointlike interactions.

In order to understand how poles in  $N = -ib/2$  can be related to large logs of  $x$  we note that the value of  $x$  determines the range of the partonic rapidity:

$$\frac{1}{2} \ln x \leq y \leq \frac{1}{2} \ln \frac{1}{x}. \quad (78)$$

It follows that as the energy increases, i.e. as  $x$  decreases, the rapidity range widens. This implies that high energy enhanced terms are not necessarily logarithmic at the level of rapidity distributions: for instance, a flat rapidity distribution leads to a contribution enhanced by  $\ln x$  in the inclusive cross-section because of this widening of the phase space. This can also be seen in Fourier-Mellin space, by noting that (see Eq. (74)) the Mellin-space inclusive cross-section is found letting  $b = 0$  in the Fourier-Mellin space rapidity distribution. Therefore poles at  $N = 0$  in the inclusive result can be obtained from any term of the form  $\left( \frac{\alpha_s}{N+f(b)} \right)^k$  such that  $f(0) = 0$ .

We now turn to terms coming from emissions from the other leg. These are easily determined: the only difference in comparison to Eq. (70) is that the boost has now parameter  $1/\bar{z}$  (see Fig. 3). The result is then identical to Eq. (77), but with the replacement:

$$\gamma \left( \frac{\alpha_s}{N + ib/2} \right) \longrightarrow \gamma \left( \frac{\alpha_s}{N - ib/2} \right). \quad (79)$$

The resummation prescription is thus the following. First, the rapidity distribution for the process  $g^*(p_L) + g^*(n_L) \rightarrow \mathcal{S}$  is determined with off-shell incoming gluons. The rules for computing  $C_y$  are the same as presented in Sect. 2.2 for the inclusive case, namely, the incoming off shell momenta are  $p_L = p + k$ ,  $p_L^2 = -k_T^2$  and  $n_L = n + \bar{k}$ ,  $n_L^2 = -\bar{k}_T^2$ , and the average over the initial gluon polarizations is performed by the projectors  $\mathcal{P}_{\mu\nu}$  Eq. (12) and  $\bar{\mathcal{P}}_{\bar{\mu}\bar{\nu}}$  Eq. (62). The coefficient function  $C_y$  is then defined using Eq. (69).

In terms of the differential coefficient function  $C_y$  the resummed rapidity distribution is given by the two leg generalization of Eq. (77):

$$\begin{aligned} \frac{d\sigma}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s \right) &= \int_{\mu^2}^{\infty} dk_T^2 \gamma \left( \frac{\alpha_s}{N + \frac{ib}{2}} \right) (k_T^2)^{\gamma(\frac{\alpha_s}{N+ib/2})-1} \times \\ &\times \int_{\mu^2}^{\infty} d\bar{k}_T^2 \gamma \left( \frac{\alpha_s}{N - \frac{ib}{2}} \right) (\bar{k}_T^2)^{\gamma(\frac{\alpha_s}{N-ib/2})-1} C_y \left( N, \frac{k_T^2}{Q^2}, \frac{\bar{k}_T^2}{Q^2}, b, \alpha_s \right). \end{aligned} \quad (80)$$

This result has been obtained with a collinear cutoff  $\mu^2$ . Although the full  $\overline{\text{MS}}$  derivation is given in Appendix B, here we give the final result:

$$\begin{aligned} \frac{d\sigma_{\overline{\text{MS}}}}{dy} (N, b, \alpha_s) &= \int_0^{\infty} d\xi \gamma \left( \frac{\alpha_s}{N + \frac{ib}{2}} \right) R_{\overline{\text{MS}}} \left( \frac{\alpha_s}{N + \frac{ib}{2}} \right) \xi^{\gamma(\frac{\alpha_s}{N+ib/2})-1} \times \\ &\times \int_0^{\infty} d\bar{\xi} \gamma \left( \frac{\alpha_s}{N - \frac{ib}{2}} \right) R_{\overline{\text{MS}}} \left( \frac{\alpha_s}{N - \frac{ib}{2}} \right) \bar{\xi}^{\gamma(\frac{\alpha_s}{N-ib/2})-1} C_y (N, \xi, \bar{\xi}, b, \alpha_s), \end{aligned} \quad (81)$$

where  $\xi = k_T^2/Q^2$ ,  $\bar{\xi} = \bar{k}_T^2/Q^2$  and we have chosen  $\alpha_s = \alpha_s(Q^2)$  and consequently dropped the explicit  $\mu^2/Q^2$  dependence.

We can cast Eq. (81) as an  $M$ -Mellin transform of the off-shell rapidity distribution  $C_y(N, \xi, \bar{\xi}, b, \alpha_s)$  in a form similar to that Eqs. (50, 51) of the inclusive case, by defining a double  $M$ -Mellin transform of the differential coefficient function  $C_y$ . Setting again  $\mu^2 = Q^2$  we get

$$h_y(N, M, \bar{M}, b, \alpha_s) \equiv M \bar{M} \int_0^{\infty} d\xi \xi^{M-1} \int_0^{\infty} d\bar{\xi} \bar{\xi}^{\bar{M}-1} C_y(N, \xi, \bar{\xi}, b, \alpha_s) R(M) R(\bar{M}), \quad (82)$$

with again  $\alpha_s = \alpha_s(Q^2)$ . The resummed result is then found from Eq. (82) through the identifications

$$\begin{aligned} M &= \gamma \left( \frac{\alpha_s}{N + \frac{ib}{2}} \right) \\ \bar{M} &= \gamma \left( \frac{\alpha_s}{N - \frac{ib}{2}} \right), \end{aligned} \quad (83)$$



namely

$$\frac{d\sigma}{dy}(N, b, \alpha_s(Q^2)) = h_y \left( N, \gamma \left( \frac{\alpha_s(Q^2)}{N + \frac{ib}{2}} \right), \gamma \left( \frac{\alpha_s(Q^2)}{N - \frac{ib}{2}} \right), b, \alpha_s(Q^2) \right). \quad (84)$$

The inclusive result (i.e. the two leg generalization of Eq. (52)) is recovered from the differential one Eq. (84) by setting  $b = 0$ . Resummed results at the hadronic level are finally obtained by substituting the resummed differential coefficient function Eq. (84) in the factorization formula Eq. (67).

As in the inclusive case, the factorized expression Eq. (81) can be viewed as a factorization theorem, by rewriting it as

$$\frac{d\sigma}{dy} = \int \frac{dz}{z} \frac{dk_T^2}{k_T^2} \int \frac{d\bar{z}}{\bar{z}} \frac{d\bar{k}_T^2}{\bar{k}_T^2} C_y \left( \frac{x}{z\bar{z}}, \frac{k_T^2}{Q^2}, \frac{\bar{k}_T^2}{Q^2}, y - \frac{1}{2} \ln \frac{z}{\bar{z}} \right) \mathcal{G}(z, k_T^2) \mathcal{G}(\bar{z}, \bar{k}_T^2). \quad (85)$$

However, a direct derivation of Eq. (85) from the determination of the gluon Green function and iteration of the BFKL kernel from its inclusive counterpart appears to be less immediate than our derivation. This is because in our case the rapidity dependence along the ladder, which has the standard  $k_T$  ordered structure, is trivial, while the rapidity flow along a BFKL ladder is less obvious.

## 4 Higgs rapidity distribution at small $x$

As a first application of the formalism developed in the previous part of the paper, we will consider Higgs production in gluon-gluon fusion. This process is particularly simple kinematically because the final state  $\mathcal{F}$  contains just one particle, the Higgs. In contrast, recalling from Sec. 2 that  $\mathcal{F}$  is the lowest-order final state for the given inclusive process with at least one incoming gluon, it is clear that for Drell-Yan [7] or prompt photon [8] production  $\mathcal{F}$  is a two-particle final state.

In the Standard Model the Higgs couples to gluons via a heavy quark loop. If the top mass  $m_{\text{top}}$  is much bigger than the Higgs mass one can describe the interaction through the effective Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{\sqrt{G_F} \sqrt{2} \alpha_s}{12\pi} H G_{\mu\nu}^a G^{\mu\nu a} \quad (86)$$

which corresponds to a pointlike interaction between the gluons and the Higgs. As it is well-known [33] this approximation is reliable within 10% for Higgs masses below 1 TeV. However, at small  $x$  this approximation fails: when the partonic center-of-mass energy squared  $s$  is large enough (i.e. at small  $x$ ) gluons can resolve the quark loop. Consequently, the effective theory at small  $x$  leads to spurious double logarithmic singularities [1, 4], while the full theory has single log behavior:

$$\sigma \sim \sigma_0 \times \begin{cases} \delta(1-x) + \sum_k c_k \alpha_s^k \ln^{2k-1} \frac{1}{x} & m_{\text{top}} \rightarrow \infty \\ \delta(1-x) + \sum_k d_k \alpha_s^k \ln^{k-1} \frac{1}{x} & \text{finite } m_{\text{top}} \end{cases} \quad (87)$$

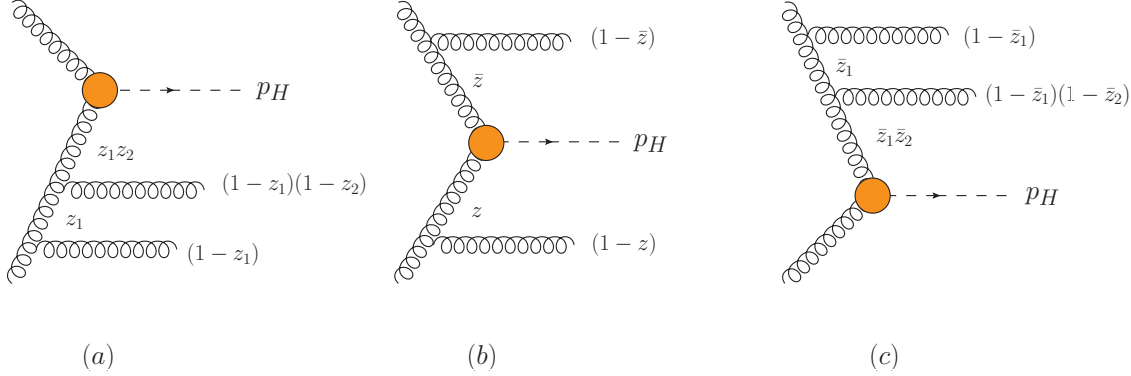


Figure 4: Graphs contributing to the small  $x$  Higgs rapidity distribution at NNLO. Here the (orange) blob stands for a  $HG_{\mu\nu}^a G^{a,\mu\nu}$  effective coupling (heavy top limit) or for the top quark loop (full Standard Model with finite  $m_{\text{top}}$ ). Only relevant light-cone fractions (in units of  $\sqrt{s/2}$ ) are reported.

The treatment of rapidity distributions for this case is particularly simple due to the fact that, using the small  $x$  kinematics Eq. (8), the rapidity is entirely fixed in terms of the momentum fractions of incoming partons. Indeed, for single emission (see Fig. 3) we have

$$y = \frac{1}{2} \ln \frac{p_H^+}{p_H^-} = \frac{1}{2} \ln \frac{1 - \frac{\bar{k}_T^2}{s(1-\bar{z})}}{\bar{z}} = \frac{1}{2} \ln \frac{1}{\bar{z}} + O\left(\frac{\bar{k}_T^2}{s}\right) + O(\bar{z}) \quad (88)$$

if the emission is from the upper leg and

$$y = \frac{1}{2} \ln \frac{z}{1 - \frac{k_T^2}{s(1-z)}} = \frac{1}{2} \ln z + O\left(\frac{k_T^2}{s}\right) + O(z) \quad (89)$$

if the emission is from the lower leg. In case of a single emission from each of the incoming legs (see Fig. 4-b), we have

$$y = \frac{1}{2} \ln \frac{z - \frac{\bar{k}_T^2}{s(1-\bar{z})}}{\bar{z} + \frac{k_T^2}{s(1-z)}} = \frac{1}{2} \ln \frac{z}{\bar{z}} + \frac{1}{2} \ln \frac{1 - \bar{\xi} \frac{x}{z\bar{z}} \bar{z}}{1 - \xi \frac{x}{z\bar{z}} z}, \quad (90)$$

with  $Q^2 = m_H^2$  (hence  $x = m_H^2/s$ ) for Higgs production. But using momentum conservation

$$\frac{x}{z\bar{z}} = \frac{m_H^2}{m_H^2 + (\mathbf{k}_T + \bar{\mathbf{k}}_T)^2} = \frac{1}{1 + \xi + \bar{\xi} + 2\sqrt{\xi\bar{\xi}} \cos \theta} \quad (91)$$

so  $\xi x/(z\bar{z})$ ,  $\bar{\xi} x/(z\bar{z})$  is at most  $O(1)$  and therefore

$$y = \frac{1}{2} \ln \frac{z}{\bar{z}} + O(z, \bar{z}). \quad (92)$$

Equations (88-92) show that for Higgs production the cross-section at fixed  $y$  is just a cross-section at fixed  $z, \bar{z}$ . Of course, these purely kinematic relations hold for any  $2 \rightarrow 1$  process.

In this section, we compute the small  $x$  behavior of the Higgs cross-section (in the  $gg$  channel) up to NNLO, before turning to the all-order resummation in the next section. In the  $m_{\text{top}} \rightarrow \infty$  limit, we will obtain analytic closed-form results which at NLO may be compared to the known [25] full analytic expression. For finite  $m_{\text{top}}$  we will reduce the NLO and NNLO small  $x$  coefficients to quadratures and tabulate them numerically as a function of  $\tau$  (i.e. of the Higgs mass, see Eq. (119)). In this case a direct comparison to the known NLO result [34] is not possible because the latter is only known in the form of the numerical integral of a fully differential cross-section.

## 4.1 Heavy top approximation

In the infinite  $m_{\text{top}}$  approximation the NLO Higgs rapidity distribution is known in closed analytic form [25]. Here we reproduce the NLO result in the small  $x$  limit. As we have mentioned in Sec. 2, only ladder-type diagrams contribute to LL $x$ . Hence we must compute the two diagrams Fig. 3. We start with emission from the upper leg. Up to power-suppressed terms in  $\bar{z}$  and  $\bar{k}_T^2/s$ , the fully differential cross-section is

$$\begin{aligned} d\bar{\sigma}_{1\text{-leg}} &= \left[ \frac{\sigma_0 \bar{z}^2 s^2}{2s\bar{z}} \times \frac{2\pi}{m_H^2} \delta \left( 1 - \frac{\bar{z}}{x} + \frac{\bar{k}_T^2}{m_H^2} \right) \right] \times \left[ \frac{\alpha_s \mu^{2\epsilon} N_c}{\pi} \frac{d\bar{z}}{\bar{z}} \frac{d\bar{k}_T^2}{(\bar{k}_T^2)^{1+\epsilon}} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right] = \\ &= \left[ \sigma_0 \frac{\bar{z}}{x} \delta \left( 1 - \frac{\bar{z}}{x} + \bar{\xi} \right) \right] \left[ \bar{\alpha}_s \frac{d\bar{z}}{\bar{z}} \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \right], \end{aligned} \quad (93)$$

where we have defined

$$\sigma_0 \equiv G_F \sqrt{2} \alpha_s^2 / 576 \pi \quad (94)$$

and in the last step we have set  $\mu^2 = Q^2 = m_H^2$  for simplicity. As in Sec. 2, the bar over  $\sigma$  indicates that it contains an unsubtracted collinear singularities, so Eq. (93) only makes sense at a regularized level.

Note that the hard part, contained in the first square bracket in Eq. (93), reduces to the LO result when  $\bar{k}_T^2 = 0$ ,  $\bar{z} = 1$ , as it ought to according to Eq. (15). The only  $\bar{k}_T$  dependence of the hard part comes from the phase space delta function, i.e. the hard amplitude is  $\bar{k}_T$  independent. This makes high energy factorization almost trivial in this case. Indeed, to compute the rapidity distribution, we just have to insert in the differential cross-section Eq. (93) the rapidity  $\delta$  function (see Eq. (68)), which in our case is

$$\begin{aligned} \delta \left( y - \frac{1}{2} \ln \frac{p_H^+}{p_H^-} \right) &= \delta \left( y - \frac{1}{2} \ln \frac{1}{\bar{z}} \right) \\ &= 2u \delta(\bar{z} - u), \end{aligned} \quad (95)$$

where, following Ref. [25]<sup>2</sup>, we have defined

$$u \equiv \exp(-2y) \quad (96)$$

and the first equality in Eq. (95) holds up to subleading terms. In terms of  $u$  the kinematic limits Eq. (78) on the rapidity become

$$x \leq u \leq \frac{1}{x}. \quad (97)$$

The transformation  $y \leftrightarrow -y$  corresponds to  $u \leftrightarrow 1/u$  symmetry. For the emission from the upper leg one obtains the  $x \leq u \leq 1$  part of the rapidity distribution (as it can be immediately seen from the equality  $u = \bar{z}$ ), while emission from the lower leg covers the other half of the rapidity range.

Using the rapidity  $\delta$  function to perform the  $\bar{z}$  integration and the momentum  $\delta$  function to perform the  $\bar{\xi}$  integration we immediately obtain the Higgs rapidity distribution as a function of  $u$ :

$$\frac{d\bar{\sigma}_{1\text{-leg}}}{du}(x, u) = \frac{1}{2u} \frac{d\sigma}{dy} = \sigma_0 \bar{\alpha}_s \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \frac{x^\epsilon}{(u-x)^{1+\epsilon}}, \quad (98)$$

where we have now omitted the explicit dependence on  $\alpha_s$  for simplicity, as well as the dependence on  $\mu^2/Q^2$  because we are working in the case  $\mu^2 = Q^2 = m_H^2$ . For  $\epsilon = 0$  Eq. (98) has an endpoint rapidity singularity in  $u = x$ . We regularize it by defining the plus-prescription

$$\begin{aligned} \int_x^1 du \frac{f(u)}{(u-x)^{1+\epsilon}} &= \int_x^1 du \frac{f(u) - f(x)}{(u-x)^{1+\epsilon}} + f(x) \int_x^1 du \frac{1}{(u-x)^{1+\epsilon}} \equiv \\ &\equiv \int_x^1 du f(x) \left[ \frac{1}{(u-x)_+} - \frac{1}{\epsilon} \delta(x-u)(1-x)^\epsilon \right], \end{aligned} \quad (99)$$

where the distribution is in  $u$  space. Using Eq. (99) and performing the  $\overline{\text{MS}}$  subtraction our result reduces to

$$\frac{d\sigma_{1\text{-leg}}}{du}(x, u) = \sigma_0 \bar{\alpha}_s \left[ \frac{1}{(u-x)_+} - \delta(u-x) \ln x \right] \quad (100)$$

up to terms which vanish like powers of  $x$ . In Eq. (100) we have taken the limit  $\epsilon \rightarrow 0$  (after  $\overline{\text{MS}}$  subtraction). Integrating Eq. (100) over  $u$ , the first term gives a vanishing contribution, while the delta function integration leads to  $\sigma = \sigma_0 \bar{\alpha}_s \ln 1/x$ , in agreement with the inclusive result [4].

One may be tempted to think that in the small  $x$  limit the first term on the right-hand side of Eq. (100) can be neglected in comparison to the second one. However, this is incorrect because the first term is phase space enhanced. To see this, we use

---

<sup>2</sup>Note that even though we follow closely the notation of Ref. [25], we depart from it in calling  $y$  the rapidity, whereas in Ref [25]  $y$  denotes a certain function of  $u$  and  $z$ .

Fourier-Mellin space. Start with the first term on the right-hand side of Eq. (100). Its Fourier transform with respect to  $u$  is

$$\int u^{-\frac{ib}{2}} \left[ \frac{1}{(u-x)_+} \right] = \frac{2i}{b} - \left( \gamma_E + i\pi + \frac{2\pi i}{e^{b\pi} - 1} \right) x^{-\frac{ib}{2}} +$$

$$-x \frac{{}_2F_1\left(1, 1 + \frac{ib}{2}, 2 + \frac{ib}{2}, x\right)}{1 + \frac{ib}{2}} - x^{-\frac{ib}{2}} \ln \frac{1}{x} - x^{-\frac{ib}{2}} \ln(1-x) - x^{-\frac{ib}{2}} \psi\left(1 - \frac{ib}{2}\right).$$
(101)

Note in particular that the  $\ln 1/x$  term in the last line exactly cancels the Fourier transform of the delta  $\delta(u-x) \ln 1/x$  contribution in Eq. (100). Performing the Mellin transform, we get

$$\mathcal{MF} \left[ \frac{1}{(u-x)_+} \right] = \frac{1 - NH(N) + NH\left(N + \frac{ib}{2}\right)}{N\left(N - \frac{ib}{2}\right)} - \frac{1}{\left(N - \frac{ib}{2}\right)^2}, \quad (102)$$

where  $H(N)$  is the  $N$ -th harmonic number. Turning now to the second term on the right-hand side of Eq. (100), its Fourier-Mellin transform is

$$\mathcal{MF} \left[ \delta(u-x) \ln \frac{1}{x} \right] = \frac{1}{\left(N - \frac{ib}{2}\right)^2}. \quad (103)$$

Combining the two terms we get

$$\mathcal{MF} \left[ \frac{1}{(u-x)_+} - \delta(u-x) \ln x \right] = \frac{1 - NH(N) + NH\left(N + \frac{ib}{2}\right)}{N\left(N - \frac{ib}{2}\right)} - \frac{1}{\left(N - \frac{ib}{2}\right)^2} +$$

$$+ \frac{1}{\left(N - \frac{ib}{2}\right)^2} = \frac{1 - NH(N) + NH\left(N + \frac{ib}{2}\right)}{N\left(N - \frac{ib}{2}\right)} = \frac{1 + O(N)}{N\left(N - \frac{ib}{2}\right)}. \quad (104)$$

The inverse Fourier-Mellin of the leading term in this result leads to the cross-section

$$\frac{d\sigma_{1\text{-leg}}}{du}(x, u) = \mathcal{MF}^{-1} \left[ \sigma_0 \bar{\alpha}_s \frac{1}{N\left(N - \frac{ib}{2}\right)} \right] = \sigma_0 \bar{\alpha}_s \frac{1}{u}, \quad (105)$$

to be compared to Eq. (100). Away from the endpoint  $x = u$ , Eq. (100) admits the small  $x$  expansion

$$\frac{d\sigma_{1\text{-leg}}}{du}(x, u) = \sigma_0 \bar{\alpha}_s \frac{1}{u-x} = \sigma_0 \bar{\alpha}_s \frac{1}{u} \left( 1 + \frac{x}{u} + \left(\frac{x}{u}\right)^2 + \dots \right), \quad (106)$$

which implies that Eq. (100) and Eq. (105) are equal up to subleading terms. Indeed, if we integrate Eq. (106) we obtain

$$\sigma_{1\text{-leg}} = \sigma_0 \bar{\alpha}_s \left( \ln \frac{1}{x} + (-x+1) + \frac{1}{2} (-x^2+1) + \dots \right). \quad (107)$$

This argument shows that at small  $x$  the  $\delta$  function in Eq. (100) cancels an analogous contribution coming from the endpoint singularity of the plus prescription, thereby demonstrating that the first term in Eq. (100) is not subleading in comparison to the second, and that small  $x$  leading terms in rapidity distributions may be unaccompanied by explicit  $\ln x$  factors. In fact, there can be nontrivial cancellations among different terms due to endpoint singularity regularization. The situation is much clearer in Fourier-Mellin-space, where we can identify all small  $x$  enhanced terms as poles in  $N = 0$  or  $N = \pm \frac{ib}{2}$ .

The emission from the other leg gives an identical contribution with only the replacement  $u \rightarrow 1/u$ , which in Fourier space amounts to  $b \rightarrow -b$ . The full two-leg NLO result in Fourier-Mellin space is thus

$$\frac{d\sigma_{\text{NLO}}}{du}(N, b) = \sigma_0 \bar{\alpha}_s \left( \frac{1}{N(N - \frac{ib}{2})} + \frac{1}{N(N + \frac{ib}{2})} \right), \quad (108)$$

which in  $x$ - $u$  space corresponds to the sum of Eq. (100) and the contribution from the other leg, obtained by letting in  $u \rightarrow 1/u$  in Eq. (100). It is easy to check by inspection that this NLO result coincides with the small  $x$  limit of the full result in [25].

We now turn to the NNLO case, for which no closed-form analytic expression exists. We must compute the three diagrams of Fig. 4, and recall that in the small  $x$  limit there are no interferences between them (see Sect. 2.6). We start by considering the case of two emissions from the same leg, say from the upper leg (see Fig. 4-c). The result for the differential cross-section is <sup>3</sup>

$$d\bar{\sigma}_{1\text{-leg}}^{2\text{-gluons}} = \left[ \sigma_0 \frac{\bar{z}_1 \bar{z}_2}{x} \delta \left( 1 - \frac{\bar{z}_1 \bar{z}_2}{x} + \bar{\xi}_1 \right) \right] \times \left[ \bar{\alpha}_s \frac{d\bar{z}_2}{\bar{z}_2} \frac{d\bar{\xi}_1}{\bar{\xi}_1^{1+\epsilon}} \right] \times \left[ \bar{\alpha}_s \frac{d\bar{z}_1}{\bar{z}_1} \frac{d\bar{\xi}_2}{(\bar{\xi}_1 + \bar{\xi}_2)^{1+\epsilon}} \right]. \quad (109)$$

The leading small  $x$  behavior in Eq. (109) comes from the ordered region  $\bar{\xi}_2 \ll \bar{\xi}_1$ ,  $\bar{z}_i \ll 1$ . With this in mind we can perform the  $\bar{\xi}_i$  integrations and the  $\overline{\text{MS}}$  subtraction to obtain

$$d\sigma_{1\text{-leg}}^{2\text{-gluons}} = \sigma_0 \bar{\alpha}_s^2 \left( \frac{1}{2} \ln^2 \frac{1}{x} \delta(\bar{z}_1 \bar{z}_2 - x) + \ln \frac{1}{x} \frac{1}{(\bar{z}_1 \bar{z}_2 - x)_+} + \left[ \frac{\ln(\bar{z}_1 \bar{z}_2 - x)}{\bar{z}_1 \bar{z}_2 - x} \right]_+ \right) d\bar{z}_1 d\bar{z}_2, \quad (110)$$

---

<sup>3</sup>From now on we will neglect all angular correlations, which can be shown to be subleading at small  $x$ . Consequently, as in the previous sections we will omit all angular terms like  $(4\pi)^\epsilon / \Gamma(1-\epsilon)$ , as they are always subtracted in the  $\overline{\text{MS}}$  scheme. See Appendix A for details.

which leads to the rapidity distribution

$$\begin{aligned}
\frac{d\sigma_{1\text{-leg}}^{2\text{-gluons}}}{du}(u, x) &= \int d\sigma_{1\text{-leg}}^{2\text{-gluons}} \delta(\bar{z}_1 \bar{z}_2 - u) = \\
&= \sigma_0 \bar{\alpha}_s^2 \left( \int_u^1 \frac{d\bar{z}_1}{\bar{z}_1} \right) \left( \frac{1}{2} \ln^2 \frac{1}{x} \delta(u-x) + \ln \frac{1}{x} \frac{1}{(u-x)_+} + \left[ \frac{\ln(u-x)}{u-x} \right]_+ \right) = \\
&= \sigma_0 \bar{\alpha}_s^2 \ln \frac{1}{u} \left( \frac{1}{2} \ln^2 \frac{1}{x} \delta(u-x) + \ln \frac{1}{x} \frac{1}{(u-x)_+} + \left[ \frac{\ln(u-x)}{u-x} \right]_+ \right). \quad (111)
\end{aligned}$$

For the sake of comparison to the resummed result, it is convenient to consider the Fourier-Mellin transform of Eq. (111). Starting from Eq. (109) and performing the  $\overline{\text{MS}}$  subtraction in Fourier-Mellin space we obtain

$$\begin{aligned}
\frac{d\sigma_{1\text{-leg}}^{2\text{-gluons}}}{dy}(N, b) &= \sigma_0 \bar{\alpha}_s^2 \frac{1}{(N - ib/2)^2} \times \\
&\times \left[ \frac{1}{2} \psi^2(N) + \gamma_E \psi(N) + \frac{1}{2} \psi^{(1)}(N) + \frac{\pi^2}{12} + \frac{\gamma_E^2}{2} \right] = \\
&= \sigma_0 \bar{\alpha}_s^2 \frac{1}{(N - ib/2)^2} \frac{1}{N^2} (1 + O(N)). \quad (112)
\end{aligned}$$

Once again, despite naive appearance, the three terms in Eq. (111) are all of the same order. The small  $x$  expansion of Eq. (111) is

$$\frac{d\sigma_{1\text{-leg}}^{2\text{-gluons}}}{du}(x, u) = \sigma_0 \bar{\alpha}_s^2 \frac{1}{u} \ln \frac{1}{u} \left( \ln \frac{1}{x} + \ln u \right) \left( 1 + O\left(\frac{x}{u}\right) \right), \quad (113)$$

or

$$\frac{d\sigma_{1\text{-leg}}^{2\text{-gluons}}}{dy}(x, y) = 8\sigma_0 \bar{\alpha}_s^2 y \left( \frac{1}{2} \ln \frac{1}{x} - y \right) (1 + O(xe^{2y})), \quad (114)$$

whose Fourier-Mellin transform is the leading- $N$  term given in the last step of Eq. (112).

Having obtained the rapidity distribution for two emissions from the upper leg (Fig. 4-c) it is immediate to obtain the result for two emissions from the lower one (Fig. 4-a), through the substitution  $y \rightarrow -y$  (i.e.  $u \rightarrow 1/u$  or  $b \rightarrow -b$ ). To complete the small  $x$  NNLO distribution we only need the distribution with one emission from each leg. In this case the leading small  $x$  term in Fourier-Mellin space reads

$$\frac{d\sigma_{2\text{-legs}}^{2\text{-gluons}}}{dy}(N, b) = 2\sigma_0 \bar{\alpha}_s^2 \frac{1}{N^2} \frac{1}{(N - ib/2)} \frac{1}{(N + ib/2)}. \quad (115)$$

Adding everything up we get

$$\frac{d\sigma_{\text{NNLO}}}{dy}(N, b) = \sigma_0 \bar{\alpha}_s^2 \frac{1}{N^2} \left( \frac{1}{N - ib/2} + \frac{1}{N + ib/2} \right)^2. \quad (116)$$

Combining the results of this section, we can write the small  $x$  rapidity distribution up to NNLO as:

$$\begin{aligned} \frac{d\sigma}{dy}(N, b) = \sigma_0 \left[ 1 + \frac{\bar{\alpha}_s}{N} \left( \frac{1}{N - ib/2} + \frac{1}{N + ib/2} \right) + \right. \\ \left. + \left( \frac{\bar{\alpha}_s}{N} \right)^2 \left( \frac{1}{N - ib/2} + \frac{1}{N + ib/2} \right)^2 + O(\alpha_s^3) \right]. \end{aligned} \quad (117)$$

Note that letting  $b = 0$  in this formula we reproduce the small  $x$  behavior of the inclusive cross-section.

## 4.2 Rapidity distribution with finite $m_{\text{top}}$

As already mentioned, the effective Lagrangian Eq. (86) is inadequate at small  $x$  and leads to spurious singularities. We now turn to the determination of the correct small  $x$  rapidity distribution.

As before we start by considering the NLO distribution, i.e. we consider single gluon emission. For definiteness, we consider first an emission from the upper leg. We obtain:

$$d\bar{\sigma}_{1\text{-leg}} = \left[ \left( \frac{\pi^3 \alpha_s^2 G_F \sqrt{2}}{4} \tau^2 |A_1(0, \bar{\xi}, \tau)|^2 \right) \frac{\bar{z}}{x} \delta \left( 1 - \frac{\bar{z}}{x} + \bar{\xi} \right) \right] \times \left[ \bar{\alpha}_s \frac{d\bar{z}}{\bar{z}} \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \right], \quad (118)$$

where

$$\tau \equiv 4m_{\text{top}}^2/m_h^2 \quad (119)$$

and  $A_1$  is the form factor defined in [5]<sup>4</sup>. Again, the hard part, i.e. the term in the first square bracket, reproduces the LO result when  $\xi \rightarrow 0$ ,  $\bar{z} \rightarrow 1$ , as expected from Eq. (15):

$$\sigma_{\text{LO}}(\tau) = \sigma_0 (12\pi^2 \tau)^2 |A_1(0, 0, \tau)|^2 \equiv \sigma_0(\tau). \quad (120)$$

Also, because

$$\lim_{\tau \rightarrow \infty} \tau A_1(\xi, 0, \tau) = \frac{1}{12\pi^2} \quad (121)$$

in the heavy top approximation Eq. (118) reproduces our previous result Eq. (93).

The form factor  $A_1$  acts as a cut-off for the  $\bar{\xi}$  integration at large  $\bar{\xi}$ , thus removing the spurious small  $x$  singularities of the effective theory. To see how this works, we consider the rapidity distribution in Fourier-Mellin space. Starting from Eq. (118) we obtain:

$$\frac{d\bar{\sigma}_{1\text{-leg}}}{dy}(N, b) = \sigma_0 (12\pi^2 \tau)^2 \left( \frac{\bar{\alpha}_s}{N - ib/2} \right) \int_0^\infty \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \frac{|A_1(0, \bar{\xi}, \tau)|^2}{(1 + \bar{\xi})^N}, \quad (122)$$

---

<sup>4</sup>Note that in Ref. [5] the definition of  $\tau$  is different.



where we have used the delta function to perform the  $N$ -Mellin transform. Without the form factor  $A_1$  (i.e. in the effective theory) the  $\bar{\xi}$  integral would be UV divergent at  $N = 0$ . This leads to a spurious  $1/N$  behavior of the integral. In the full theory this problem is no longer there: the integral is cut off by  $A_1$ , hence at small  $x$  we can safely let  $N = 0$ :

$$\frac{d\bar{\sigma}_{1\text{-leg}}}{dy}(N, b) = \sigma_0(12\pi^2\tau)^2 \left( \frac{\bar{\alpha}_s}{N - ib/2} \right) \int_0^\infty \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} |A_1(0, \bar{\xi}, \tau)|^2. \quad (123)$$

The  $\bar{\xi}$  integral now is well-behaved in the ultraviolet, but it still contains a collinear singularity. To extract it, we simply integrate by parts:

$$\begin{aligned} \int_0^\infty \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} |A_1(0, \bar{\xi}, \tau)|^2 &= \\ &= -\frac{1}{\epsilon} \left\{ \left[ |A_1(0, \bar{\xi}, \tau)|^2 \bar{\xi}^{-\epsilon} \right]_{\bar{\xi}=0}^{\bar{\xi} \rightarrow \infty} - \int_0^\infty d\bar{\xi} \bar{\xi}^{-\epsilon} \frac{d|A_1(0, \bar{\xi}, \tau)|^2}{d\bar{\xi}} \right\} \\ &= -\frac{1}{\epsilon} \left\{ |A_1(0, 0, \tau)|^2 + \epsilon \int_0^\infty d\bar{\xi} \ln \bar{\xi} \frac{d|A_1(0, \bar{\xi}, \tau)|^2}{d\bar{\xi}} + O(\epsilon^2) \right\}, \end{aligned} \quad (124)$$

where we have used

$$\lim_{\bar{\xi} \rightarrow 0, \infty} |A_1(0, \bar{\xi}, \tau)|^2 \bar{\xi}^{-\epsilon} = 0 \quad (125)$$

since  $\epsilon < 0$ . We can thus write the  $\overline{\text{MS}}$  subtracted rapidity distribution as

$$\frac{d\sigma_{1\text{-leg}}}{dy}(N, b) = \sigma_0(\tau) \frac{\bar{\alpha}_s}{N - ib/2} c_1(\tau), \quad (126)$$

where we have defined

$$c_1(\tau) \equiv -\frac{1}{|A_1(0, 0, \tau)|^2} \int_0^\infty d\bar{\xi} \ln \bar{\xi} \frac{d|A_1(0, \bar{\xi}, \tau)|^2}{d\bar{\xi}}. \quad (127)$$

Numerical results for  $c_1(\tau)$  are tabulated in Tab. 1. Note that the result Eq. (126) does not depend on  $N$  and  $b$  separately, but only on the combination  $N - ib/2$ . This implies that in  $(x, y)$  space the result has a rapidity delta function:

$$\frac{d\sigma_{1\text{-leg}}}{dy}(x, y) = \sigma_0(\tau) c_1(\tau) \bar{\alpha}_s \delta \left( y - \frac{1}{2} \ln \frac{1}{x} \right). \quad (128)$$

To obtain the result for emission from the lower leg it is sufficient to symmetrize this result. The full small  $x$  NLO result thus is

$$\frac{d\sigma_{\text{NLO}}}{dy}(x, y) = \sigma_0(\tau) c_1(\tau) \bar{\alpha}_s \left[ \delta \left( y - \frac{1}{2} \ln \frac{1}{x} \right) + \delta \left( y + \frac{1}{2} \ln \frac{1}{x} \right) \right], \quad (129)$$

or in Fourier-Mellin space

$$\frac{d\sigma_{\text{NLO}}}{dy}(N, b) = \sigma_0(\tau)c_1(\tau)\bar{\alpha}_s \left( \frac{1}{N - ib/2} + \frac{1}{N + ib/2} \right). \quad (130)$$

Note that the small  $x$  behavior of the rapidity distribution with finite top mass is completely different from that in the effective theory: in the former case the rapidity is forced to sit at its endpoints, while in the latter we have a flat rapidity distributions (see Eq. (105) and recall  $dy = du/(2u)$ ). Note also that the integration of Eq. (129) leads to a constant, in contrast with the spurious  $\ln x$  behavior of the effective theory.

We can compare also the NNLO distributions. As in the previous subsection, we start from two emissions from the upper leg. In this case the small  $x$  rapidity distribution in Fourier-Mellin space is

$$\frac{d\bar{\sigma}_{1\text{-leg}}^{2\text{-gluons}}}{dy}(N, b) = \sigma_0(12\pi^2\tau)^2 \left( \frac{\bar{\alpha}_s}{N - ib/2} \right)^2 \int_0^\infty \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \frac{|A_1(0, \bar{\xi}, \tau)|^2}{(1 + \bar{\xi})^N} \left( -\frac{1}{\epsilon} \frac{1}{\bar{\xi}^\epsilon} + \frac{1}{\epsilon} \right), \quad (131)$$

where we have regulated just one of the two collinear singularities coming from the extra gluon emission. To deal with Eq. (131) we first note that we can safely let  $N = 0$  in the  $\bar{\xi}$  integral as before. To deal with the remaining collinear singularity, we just integrate by parts as in the NLO case. The  $\overline{\text{MS}}$  result is then

$$\frac{d\sigma_{1\text{-leg}}^{2\text{-gluons}}}{dy}(N, b) = \sigma_0(\tau)c_2(\tau) \left( \frac{\bar{\alpha}_s}{N - ib/2} \right)^2, \quad (132)$$

with

$$c_2(\tau) \equiv -\frac{1}{|A_1(0, 0, \tau)|^2} \int_0^\infty d\bar{\xi} \frac{\ln^2 \bar{\xi}}{2} \frac{d|A_1(0, \bar{\xi}, \tau)|^2}{d\bar{\xi}}. \quad (133)$$

Numerical results for  $c_2$  are tabulated in Tab. 1.

The results with two emissions from the lower leg is trivially obtained by performing the replacement  $b \rightarrow -b$  in Eq. (132). In the nontrivial case of one emission from each leg the differential cross-section is

$$\begin{aligned} d\bar{\sigma}_{2\text{-legs}}^{2\text{-gluons}} &= \left[ \sigma_0(12\pi^2\tau)^2 2 \left| A_1(\xi, \bar{\xi}, \tau) \cos \theta + A_3(\xi, \bar{\xi}, \tau) \sqrt{\xi\bar{\xi}} \right|^2 \frac{\bar{z}z}{x} \times \right. \\ &\times \delta \left( 1 - \frac{\bar{z}z}{x} + \xi + \bar{\xi} + 2\sqrt{\xi\bar{\xi}} \cos \theta \right) \frac{d\theta}{2\pi} \left[ \bar{\alpha}_s \frac{d\bar{z}}{\bar{z}} \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \right] \left[ \bar{\alpha}_s \frac{dz}{z} \frac{d\xi}{\xi^{1+\epsilon}} \right], \end{aligned} \quad (134)$$

where  $A_3$  was defined in Ref. [5].

Note that if we let  $\xi = 0$  in the first square bracket of Eq. (134) and perform the angular integration we recover the one-leg result Eq. (118). From the differential

$\tau$	$c_1$	$c_2$	$c_{1,1}$
1.5000	0.4869	1.6119	0.7752
2.0000	0.8372	1.8540	1.2129
2.5000	1.0923	2.1056	1.6925
3.0000	1.2942	2.3500	2.1670
3.5000	1.4616	2.5831	2.6237
4.0000	1.6046	2.8042	3.0596
4.5000	1.7297	3.0141	3.4746
5.0000	1.8407	3.2134	3.8699
5.5000	1.9406	3.4031	4.2470
6.0000	2.0314	3.5841	4.6073
6.5000	2.1146	3.7571	4.9522
7.0000	2.1914	3.9230	5.2831
7.5000	2.2627	4.0822	5.6011
8.0000	2.3292	4.2353	5.9073
8.5000	2.3916	4.3829	6.2026
9.0000	2.4502	4.5253	6.4877
9.5000	2.5058	4.6630	6.7635
10.0000	2.5582	4.7963	7.0306
10.5000	2.6081	4.9255	7.2896
11.0000	2.6556	5.0509	7.5411
11.5000	2.7010	5.1724	7.7854
12.0000	2.7444	5.2908	8.0231

Table 1: Numerical values for the NLO coefficient  $c_1$  Eq. (127) and for the NNLO coefficients  $c_2$  Eq. (133) and  $c_{1,1}$  Eq. (137).

cross-section Eq. (134) we can easily write down the rapidity distribution in Fourier-Mellin space

$$\begin{aligned} \frac{d\bar{\sigma}_{2\text{-legs}}^{2\text{-gluons}}}{dy}(N, b) &= \sigma_0(12\pi^2\tau)^2 \left( \frac{\bar{\alpha}_s}{N - ib/2} \right) \left( \frac{\bar{\alpha}_s}{N + ib/2} \right) \times \\ &\times \int \frac{d\xi}{\xi^{1+\epsilon}} \frac{d\bar{\xi}}{\bar{\xi}^{1+\epsilon}} \frac{|A_1(\xi, \bar{\xi}, \tau) \cos \theta + A_3(\xi, \bar{\xi}, \tau) \sqrt{\xi \bar{\xi}}|^2}{(1 + \xi + \bar{\xi} + 2\sqrt{\xi \bar{\xi}} \cos \theta)^N} \frac{d\theta}{\pi}. \end{aligned} \quad (135)$$

As in the previous case we can let  $N = 0$  in the integral and perform the  $\overline{\text{MS}}$  subtraction by integrating by parts. The final subtracted result now reads

$$\frac{d\sigma_{2\text{-legs}}^{2\text{-gluons}}}{dy}(N, b) = \sigma_0(\tau) c_{1,1}(\tau) \left( \frac{\bar{\alpha}_s}{N - ib/2} \right) \left( \frac{\bar{\alpha}_s}{N + ib/2} \right) \quad (136)$$

with

$$c_{1,1}(\tau) \equiv \frac{1}{|A_1(0, 0, \tau)|^2} \int_0^\infty d\xi \int_0^\infty d\bar{\xi} \left[ \ln \xi \ln \bar{\xi} \frac{\partial^2 |A_1(\xi, \bar{\xi}, \tau)|^2}{\partial \xi \partial \bar{\xi}} + 2 |A_3(\xi, \bar{\xi}, \tau)|^2 \right]. \quad (137)$$

Numerical values for  $c_{1,1}$  are tabulated in Tab. 1.

Adding up all contribution up to NNLO we obtain the following small  $x$  rapidity distribution:

$$\begin{aligned} \frac{d\sigma}{dy}(N, b) &= \sigma_0(\tau) \left\{ 1 + c_1(\tau) \bar{\alpha}_s \left[ \frac{1}{N - ib/2} + \frac{1}{N + ib/2} \right] \right. \\ &+ \bar{\alpha}_s^2 \left[ c_2(\tau) \left( \left( \frac{\bar{\alpha}_s}{N - ib/2} \right)^2 + \left( \frac{\bar{\alpha}_s}{N + ib/2} \right)^2 \right) \right. \\ &\left. \left. + c_{1,1}(\tau) \left( \frac{\bar{\alpha}_s}{N - ib/2} \right) \left( \frac{\bar{\alpha}_s}{N + ib/2} \right) \right] + O(\alpha_s^3) \right\}. \end{aligned} \quad (138)$$

## 5 Resummation of the Higgs rapidity distribution

In this section we use Eq. (81) (or, equivalently, Eq. (84)) to perform the small  $x$  resummation of the Higgs rapidity distribution, both in the effective theory and for finite top mass. We will check that that up to NNLO the resummed results agree with the explicit computations presented in the previous section.

### 5.1 Pointlike effective interaction

The resummation of the inclusive cross-section in the limit  $m_{\text{top}} \rightarrow \infty$  was first performed in Ref. [4]. The resummed result for the gluon-gluon sub-process in

Mellin space is

$$\sigma_{gg}(N) = \frac{\sigma_0}{1 - \frac{2\gamma}{N}} R_{\overline{\text{MS}}}^2(\gamma) = R_{\overline{\text{MS}}}^2(\gamma) \sigma_0 \left( 1 + \frac{2\gamma}{N} + \left( \frac{2\gamma}{N} \right)^2 + \dots \right) \quad (139)$$

where  $\gamma = \gamma\left(\frac{\alpha_s}{N}\right)$  is the LL $x$  anomalous dimension. Because of the simple kinematic relation (92) between  $z$  and  $y$ , the only further piece of information needed for the resummation is the off-shell cross-section with the contributions from emissions coming from the two legs kept separate. This is given by [4]

$$\sigma_{gg}(N) = \frac{\sigma_0}{1 - M - \bar{M}} R_{\overline{\text{MS}}}(M) R_{\overline{\text{MS}}}(\bar{M}) \Bigg|_{M=\bar{M}=\gamma\left(\frac{\alpha_s}{N}\right)}, \quad (140)$$

where the two variables  $M$ ,  $\bar{M}$  correspond to radiation from either leg. Using Eq. (84), we immediately get

$$\frac{d\sigma_{gg}}{dy}(N, b) = \frac{\sigma_0 R_{\overline{\text{MS}}}\left(\gamma\left(\frac{\alpha_s}{N-ib/2}\right)\right) R_{\overline{\text{MS}}}\left(\gamma\left(\frac{\alpha_s}{N+ib/2}\right)\right)}{1 - \frac{1}{N}\gamma\left(\frac{\alpha_s}{N-ib/2}\right) - \frac{1}{N}\gamma\left(\frac{\alpha_s}{N+ib/2}\right)}. \quad (141)$$

Note that, in accordance with Eq. (84), the argument  $N$  in the anomalous dimension is shifted by  $\pm ib$ , but the intrinsic  $N$  dependence of the coefficient function is not.

Recalling that  $R_{\overline{\text{MS}}} = O(\alpha_s^3)$  it is immediate to check that the resummed result agrees with the fixed-order results up to NNLO obtained in Sect. 4.1 Eq. (117).<sup>5</sup> Expanding Eq. (141) up to NNLO and performing the inverse Fourier-Mellin transform, the result in terms of the partonic rapidity  $y$  is

$$\begin{aligned} \frac{d\sigma_{gg}}{dy}(x, y) = & \sigma_0 \left\{ \frac{\delta(1-x)}{2} \left[ \delta\left(y - \frac{1}{2} \ln \frac{1}{x}\right) + \delta\left(y - \frac{1}{2} \ln x\right) \right] \right. \\ & \left. + 2\bar{\alpha}_s + 4\bar{\alpha}_s^2 \left( \frac{\ln^2 x}{4} - y^2 \right) + O(\alpha_s^3) \right\}. \end{aligned} \quad (142)$$

## 5.2 Finite top mass

We now consider the case of finite top mass. The resummation of the inclusive cross-section has been performed in Ref. [5]. The result was written in terms of a double Mellin transform

$$\begin{aligned} \sigma_{gg}(N) = & \sigma_0 (12\pi^2)^2 \tau^2 R_{\overline{\text{MS}}}(M) R_{\overline{\text{MS}}}(\bar{M}) M \bar{M} \times \\ & \times \int_0^\infty d\xi \xi^{M-1} \int_0^\infty d\bar{\xi} \bar{\xi}^{\bar{M}-1} [|A_1|^2 + 2\xi\bar{\xi} |A_3|^2] \Bigg|_{M=\bar{M}=\gamma\left(\frac{\alpha_s}{N}\right)}, \end{aligned} \quad (143)$$

---

<sup>5</sup>It follows that these results cannot be used to check a recent claim [35] that the standard result of  $R$  from Refs. [1, 19] is incorrect.

where as above  $M$  and  $\bar{M}$  correspond to radiation from either of the two legs. Expanding Eq. (143) in powers of  $M$  and  $\bar{M}$  one obtains the coefficient of the  $LLx$  singularity to any desired order in terms of double integrals over  $\xi$  and  $\bar{\xi}$ , which have to be evaluated numerically. Note that now in Eq. (143) all the  $N$  dependence comes from  $\gamma(\alpha_s/N)$ , since in the small  $x$  limit the form factors  $A_i$  are  $N$ -independent as one expects for a non-pointlike interaction.

It is immediate to obtain the rapidity distribution:

$$\begin{aligned} \frac{d\sigma_{gg}}{dy}(N, b) &= \sigma_0(12\pi^2)^2 \tau^2 R_{\overline{\text{MS}}}(M_b) R_{\overline{\text{MS}}}(\bar{M}_b) M_b \bar{M}_b \\ &\int_0^\infty d\xi \xi^{M_b-1} \int_0^\infty d\bar{\xi} \bar{\xi}^{\bar{M}_b-1} [|A_1|^2 + 2\xi\bar{\xi}|A_3|^2], \end{aligned} \quad (144)$$

where we have defined

$$M_b \equiv \gamma\left(\frac{\alpha_s}{N - \frac{ib}{2}}\right); \quad \bar{M}_b \equiv \gamma\left(\frac{\alpha_s}{N + \frac{ib}{2}}\right). \quad (145)$$

Integrating Eq. (144) by parts we can make its perturbative expansion explicit:

$$\begin{aligned} \frac{d\sigma_{gg}}{dy}(N, b) &= \sigma_0(\tau) R_{\overline{\text{MS}}}(M_b) R_{\overline{\text{MS}}}(\bar{M}_b) \times \\ &\times \left[ \sum_{i \geq 0} (M_b^i + \bar{M}_b^i) c_i(\tau) + \sum_{j, k > 0} M_b^j \bar{M}_b^k c_{j,k}(\tau) \right] \end{aligned} \quad (146)$$

with

$$\begin{aligned} c_i(\tau) &\equiv -\frac{1}{|A_1(0, 0, \tau)|^2} \int_0^\infty d\xi \frac{\ln^i \xi}{i!} \frac{d|A_1(\xi, 0, \tau)|^2}{d\xi} \\ c_{j,k}(\tau) &\equiv \frac{1}{|A_1(0, 0, \tau)|^2} \int_0^\infty d\xi \int_0^\infty d\bar{\xi} \left[ \frac{\ln^j \xi}{j!} \frac{\ln^k \bar{\xi}}{k!} \frac{\partial^2 |A_1(\xi, \bar{\xi}, \tau)|^2}{\partial \xi \partial \bar{\xi}} \right. \\ &\quad \left. + 2 \frac{\ln^{j-1} \xi}{(j-1)!} \frac{\ln^{k-1} \bar{\xi}}{(k-1)!} |A_3(\xi, \bar{\xi}, \tau)|^2 \right]. \end{aligned} \quad (147)$$

Integrating Eq. (146) we obtain the right single logarithmic behavior of the total cross-section.

For comparison with the explicit results of the previous section, we write the  $(x, y)$  resummed rapidity distribution up to NNLO:

$$\begin{aligned} \frac{d\sigma_{gg}}{dy}(x, y) &= \sigma_0(\tau) \left\{ \frac{\delta(1-x)}{2} \left[ \delta\left(y - \frac{1}{2} \ln x\right) + \delta\left(y + \frac{1}{2} \ln x\right) \right] + \right. \\ &\quad \left. + \bar{\alpha}_s c_1(\tau) \left[ \delta\left(y - \frac{1}{2} \ln x\right) + \delta\left(y + \frac{1}{2} \ln x\right) \right] + \right. \\ &\quad \left. + \bar{\alpha}_s^2 \left[ \left[ \delta\left(y - \frac{1}{2} \ln x\right) + \delta\left(y + \frac{1}{2} \ln x\right) \right] \ln \frac{1}{x} c_2(\tau) + c_{1,1}(\tau) \right] \right\}, \end{aligned} \quad (148)$$

which agrees with our previous findings. The pattern of Eq. (148) persists at higher orders: there are two endpoint contributions coming from emissions from just one leg, plus a bulk term coming from emissions from both legs.

So far we have only considered the gluon-gluon partonic sub-process. However, beyond LO other channels open up. As discussed in Sect. 2.6, the high energy behavior of the quark initiated sub-processes can be straightforwardly computed from the gluon-gluon case thanks to color-charge relations. The LL $x$  behavior for the inclusive Higgs production in the different partonic channels have been explicitly computed to NNLO in [9]. The generalization to the rapidity distributions is not difficult. Up to NNLO we have

$$\begin{aligned} \frac{d\sigma_{gg}}{dy}(x, y) = \sigma_0(\tau) \frac{C_F}{C_A} & \left\{ \bar{\alpha}_s \left[ \delta \left( y - \frac{1}{2} \ln x \right) + \delta \left( y + \frac{1}{2} \ln x \right) \right] \frac{c_1(\tau)}{2} + \right. \\ & \left. + \bar{\alpha}_s^2 \left[ \delta \left( y - \frac{1}{2} \ln x \right) + \delta \left( y + \frac{1}{2} \ln x \right) \right] \ln \frac{1}{x} \frac{c_2(\tau)}{2} + c_{1,1}(\tau) + O(\alpha_s^3) \right\}; \end{aligned} \quad (149)$$

$$\frac{d\sigma_{q_i q_j (\bar{q}_j)}}{dy}(x, y) = \sigma_0(\tau) \left\{ \bar{\alpha}_s^2 \left( \frac{C_F}{C_A} \right)^2 c_{1,1}(\tau) + O(\alpha_s^3) \right\}. \quad (150)$$

### 5.3 Matching to the effective theory

As mentioned in the introduction, resummed results are useful not only for the sake of all-order resummed phenomenology, but also as a way of obtaining information on higher order perturbative terms. This is particularly interesting in the case of Higgs production, where results are most easily obtained in the  $m_{\text{top}} \rightarrow \infty$  limit, which however fails at small  $x$ . It is then possible to improve the  $m_{\text{top}} \rightarrow \infty$  result by correcting its small  $x$  behavior with its exact finite  $m_{\text{top}}$  form extracted from resummation. For the inclusive NNLO cross-section this was done in Ref. [6]. A more ambitious task is to construct an accurate approximation to the full finite  $m_{\text{top}}$  result by matching the small  $x$  behavior from resummation to an expansion in powers of  $1/m_{\text{top}}$ , which at the inclusive NNLO level was done in Ref. [9].

Here we perform a matched determination akin to that of Ref. [6] for the NLO rapidity distribution, by combining the small  $x$  behavior Eqs. (129, 149) with the analytic expression in the  $m_{\text{top}} \rightarrow \infty$  limit from Ref. [25]. This quantity has also been obtained by numerical integration of a fully differential expression in Ref. [27], with which we will compare our results. In Ref. [25] the rapidity distribution is parametrized by the variables  $z$  and  $y$  where  $z$  coincides with the variable that we have called  $x$  throughout this paper (compare in particular Eq. (68)) while  $y$  should not be confused with the rapidity, and coincides with the variable  $w$  defined as

$$w \equiv \frac{u - x}{(1 - x)(1 + u)} \quad (151)$$

in terms of  $x$  and  $u$  Eq. (96).

The matched result is constructed as follows:

$$\frac{d\sigma_{ij}}{dw} = \begin{cases} \left. \frac{d\sigma_{ij}}{dw} \right|_{\text{eff}} & x > x_{\text{match}} \\ \left. \frac{d\sigma_{ij}}{dw} \right|_{\text{eff}} + \left. \frac{d\sigma_{ij}}{dw} \right|_{\text{matching}} & x \leq x_{\text{match}} \end{cases} \quad (152)$$

where  $ij=gg$  or  $qg$ , by  $\left. \frac{d\sigma_{ij}}{dw} \right|_{\text{eff}}$  we denote the result in the heavy top limit as given in Eqs. (24)-(25) of Ref. [25], and  $\left. \frac{d\sigma_{ij}}{dw} \right|_{\text{matching}}$  is a matching term which subtracts the spurious double log small  $x$  behavior from the heavy top result, and it replaces it with the correct small  $x$  behavior for finite  $m_{\text{top}}$ .

The matching term in turn is determined by noting that at NLO at the inclusive level the spurious double logs correspond to terms which in Mellin space behave as either a simple or a double  $N$  pole (i.e. as functions of  $x$  which are either constant or grow as  $\ln x$  as  $x \rightarrow 0$ ). Thus, the matching terms are

$$\begin{aligned} \left( \frac{d\sigma_{gg}}{dw} \right)_{\text{matching}} &= - \left[ 3 \frac{x^2 - 1}{[w(x-1)+1][w(x-1)-x]} - 3(2 - w(1-w)) \right] + \\ &+ 3c_1(\tau) [\delta(1-w) + \delta(w)], \end{aligned} \quad (153)$$

$$\begin{aligned} \left( \frac{d\sigma_{qg}}{dw} \right)_{\text{matching}} &= - \left[ 2 \frac{x^2 - 1}{[w(x-1)+1][w(x-1)-x]} - 1 \right] + \\ &+ 2c_1(\tau) [\delta(1-w) + \delta(w)]. \end{aligned} \quad (154)$$

where the first line of Eqs. (153,154) subtracts all contributions to the heavy top results (respectively Eq. (25) and Eq. (24) of Ref. [25]) which upon integration lead to contributions to the inclusive result which either grow or go to a constant at small  $x$ . The second line of both expressions adds back the correct single-log behavior as given respectively in Eq. (129), and Eq. (149). Note that the first term on the right-hand side of Eqs. (153, 154) is the leading double-log contribution computed in Sect. 5.1, Eq. (106), symmetrized, expressed in terms of  $w$  and multiplied by an appropriate Jacobian; when expressed in terms of the rapidity  $y$  this contribution is a flat rapidity distribution, as already noticed in Eq. (130). The second subtraction term on the right-hand side of Eqs. (153, 154) is  $\text{NLL}x$  hence it is not determined by  $\text{LL}x$  resummation and we have extracted it from Ref. [25] directly. We will choose for  $x_{\text{match}}$  the same matching value as used in Refs. [5, 6]. This ensures that our result reproduces the inclusive one once it has been integrated over the rapidity range.

We have computed the full hadronic rapidity distribution up to NLO, using for the NLO term the matched results Eqs. (152-154). The hadron-level result is given by

$$\frac{d\sigma}{dY} = \int_0^1 dw \int_{x_h}^1 dx \frac{d\sigma}{dw} \mathcal{L}(x, w, Y), \quad (155)$$



where the partonic luminosity  $\mathcal{L}$  is defined as

$$\mathcal{L}(x, w, Y) \equiv \frac{x_h}{x^2} f\left(\frac{\sqrt{x_h} e^Y}{\sqrt{x/u}}\right) f\left(\frac{\sqrt{x_h} e^{-Y}}{\sqrt{x/u}}\right), \quad (156)$$

and  $u$  is obtained by inverting Eq. (151). Results for two different values of the Higgs mass ( $m_h = 130$  GeV and  $m_h = 280$  GeV), at the LHC with  $\sqrt{S} = 7$  TeV and  $\sqrt{S} = 14$  TeV are shown in Fig. 5; we have used NNPDF2.0 parton distributions [36] (PDF uncertainties are not shown). We plot the ratio of the NLO matched result to the large top mass result of Ref. [25], divided by the corresponding ratio of total cross-sections, so that only shape differences are shown in the plot. These plots confirm the conclusion of Ref. [27] that corrections to the NLO rapidity distribution due to finite-mass effects are below 5%. We also determine the precise shape of the correction, which could not be determined in Ref [27] because of insufficient numerical accuracy. In comparison to that reference, we also seem to find a somewhat more noticeable correction in the central rapidity region, though of course the effect on the total cross-section remains quite small. For heavier Higgs masses the correction to the shape becomes smaller.

More refined matching procedures might be constructed, for example by taking the value of  $x_{\text{match}}$  to depend on  $y$ . However, the subtraction term (first line on the right-hand side of Eqs. (153, 154) turns out to have a very small impact on the shape correction, at least for the values of the Higgs mass and energy considered here. This can be understood in part as a consequence of the fact that the leading double logarithmic term (first term on the right-hand side of Eqs. (153, 154) is flat in rapidity, as mentioned above, so it does not affect the shape.

A similar matched procedure cannot be carried out to NNLO because an analytic expression of the Higgs rapidity distribution to NNLO does not exist even in the heavy top limit, to the best of our knowledge. It is possible nevertheless to provide a rough estimate (more likely an upper bound) of the impact of finite top mass corrections on the NNLO cross-section by computing the  $K$ -factor

$$K = \left( \frac{d\sigma^{\text{LO}}}{dY} + \frac{d\sigma^{\text{NLO}}}{dY} + \frac{d\sigma_{\text{LL}x}^{\text{NNLO}}}{dY} \right)_{\text{matched}} \bigg/ \left( \frac{d\sigma^{\text{LO}}}{dY} + \frac{d\sigma^{\text{NLO}}}{dY} + \frac{d\sigma_{\text{LL}x}^{\text{NNLO}}}{dY} \right)_{\text{eff}} \quad (157)$$

where at NNLO level only the contribution from the small  $x$  tail is included, in turn approximated in each case by its LL $x$  behavior, namely

$$\frac{d\sigma_{\text{LL}x}^{\text{NNLO}}}{dY} \equiv \int_0^1 dw \int_{x_h}^{x_{\text{match}}} dx \frac{d\sigma_{\text{LL}x}^{\text{NNLO}}}{dw} \mathcal{L}(x, w, Y) \quad (158)$$

with  $d\sigma_{\text{LL}x}^{\text{NNLO}}/dw$  given by the NNLO term in Eq. (148) for the matched case and by the NNLO term of Eq. (142) for the effective (heavy top) case, in each case multiplied by the appropriate Jacobian. We found that at the LHC at 7 TeV the finite mass corrections are well below 1%, while at 14 TeV they are larger but still below 2% in the full rapidity range. This suggests that the use of the effective theory is fully justified for rapidity distributions also beyond NLO.

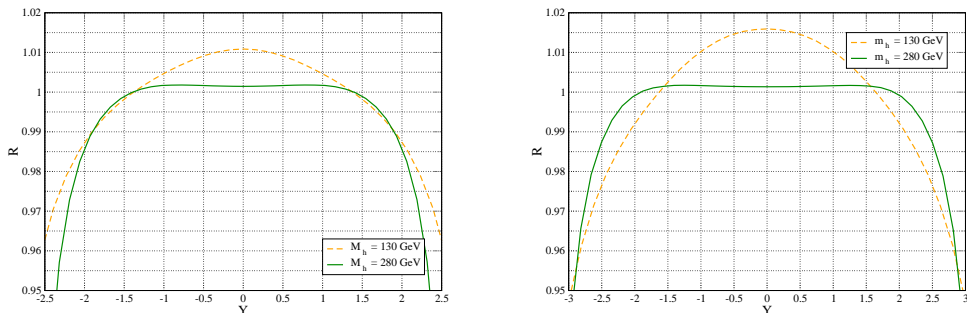


Figure 5: The ratio of the rapidity distribution for Higgs production computed up to NLO using the matched expressions Eqs. (152-154) to the result in the heavy top limit of Ref. [25]. The ratio is rescaled by the corresponding ratio of total cross-sections. Results are shown for the LHC at 7 TeV (left) and 14 TeV (right); the dashed yellow curves are for  $m_h = 130$  GeV and the solid green ones for  $m_h = 280$  GeV.

## 6 Conclusions and Outlook

In this paper we have accomplished for the first time small  $x$  resummation of a differential cross-section (as opposed to a fully inclusive one). This has been done by extending to differential rapidity distributions the so-called high energy or  $k_T$  factorization, which expresses  $LLx$  cross-sections in terms of standard partonic cross-sections, but computed with incoming off-shell gluons. The result has been made possible by exploiting the duality which relates  $LLx$  and  $LLQ^2$  evolution equations, to re-express high energy factorization in terms of standard collinear factorization, which in turns is straightforwardly extended to rapidity distributions.

Our final resummation prescription is quite close to the well-established one derived long ago for inclusive cross-sections [1]. Indeed, it is based on evaluating the LO rapidity distribution for the relevant process but with incoming off-shell gluons. As in the inclusive case, the leading high energy behavior is found by means of a pole approximation in Mellin space, whereby the Mellin variable which is conjugate to the gluon virtuality is identified with the  $LLx$  anomalous dimension. However, one must now also take a Fourier transform with respect to rapidity, and the argument of  $LLx$  anomalous dimensions undergoes an impact-parameter dependent shift in the complex plane of the variable which is Mellin conjugate to the scaling variable  $x$ .

As first application, we have performed a resummation of the rapidity distribution for Higgs production in gluon-gluon fusion. We have computed the small  $x$  limit of the rapidity distribution to NNLO, both in the heavy top mass approximation and for finite  $m_{\text{top}}$  (only the  $m_{\text{top}} \rightarrow \infty$  NLO result being already available in

closed form). We have shown the way these expressions are related to the results obtained by expanding to finite order the resummation formula and inverting the Fourier-Mellin transform, and we have seen that this involves nontrivial cancellations between terms which at the resummed level have different origins, thereby providing a nontrivial consistency check on the resummation procedure. As a first example of application we have constructed an approximate analytic expression for the NLO rapidity distribution with finite top mass, which appears to be in agreement with the numerical results of Ref. [27], but provides a prediction which is not affected by problems of numerical accuracy.

The potentially more interesting application of this formalism is to Drell-Yan rapidity distributions, which will be explored at very small  $x$  at the LHC, specifically by the LHCb collaboration [18], and which may play an important role both in determining parton distributions and testing resummed QCD predictions at small  $x$ . More in general, our result provide a first generalization of small  $x$  factorization and resummation beyond the inclusive  $LLx$  level. It will be interesting to see whether they can be generalized not only to less inclusive quantities, but also beyond the leading logarithmic approximation.

**Acknowledgment** We thank R. D. Ball for several discussions which are at the origin of this paper and for a critical reading of the manuscript. F.C. acknowledges many useful discussions with G. Diana, S.F. thanks V. del Duca for discussions during the early stage of this work, and S.M. thanks F. Petriello for fruitful correspondence on the fixed-order results. This work was supported in part by the European network HEPTOOLS under contract MRTN-CT-2006-035505 and by an Italian PRIN-2008 grant. The work of S.M. is supported by UK's STFC.

## Appendix

### A The $\overline{\text{MS}}$ subtraction procedure to all orders

Here we briefly sketch how to subtract collinear singularities arising from kernel iteration within the  $\overline{\text{MS}}$  scheme. Here we outline only the features which are relevant to understand how Eqs. (31), (42) are obtained. For a general treatment, including a discussion of ultraviolet and infrared singularities, as well as for a more detailed explanation, we refer the reader to the original papers [30, 21].

We start by writing a generic cross-section as the convolution product of a “bare” (i.e. unsubtracted) coefficient function  $\bar{M}$  and a “bare” PDF  $\bar{\Gamma}$ :

$$\sigma = \bar{M} \otimes \bar{\Gamma}, \quad (159)$$

where  $\otimes$  denotes the standard convolution product in  $x$  space. We assume to work in  $d = 4 - 2\epsilon$  dimensions in order to regularize  $\bar{M}$  and  $\bar{\Gamma}$ . Following Sect. 2, we now write the coefficient function  $\bar{M}$  as the product of a hard part  $H$  and the iteration of a kernel  $K$ :

$$\bar{M} = H \otimes_{x, k_T} (1 + K + K \otimes_{x, k_T} K + \dots) \equiv M(1 + K + K^2 + K^3 + \dots) = H \frac{1}{1 - K}, \quad (160)$$

where now the convolution  $\otimes_{x, k_T}$  stands both for normal convolution in  $x$  space and for  $k_T$  integration in the transverse space. This convolution product will be understood in the following formal manipulations.

As a result of the  $k_T$  integrations, the coefficient function Eq. (160) contains multiple collinear poles of the form  $1/\epsilon^j$  and thus it is meaningless in 4 dimensions. To perform the subtraction, we first introduce the “pole-part” projector  $P$  which acting on a generic function  $F$  selects just its  $1/\epsilon^k$  poles, i.e.

$$PF \equiv \sum_{k>0} \frac{1}{\epsilon^k} \lim_{\epsilon \rightarrow 0} (\epsilon^k F). \quad (161)$$

We now use this projector to factorize the kernel  $1/(1 - K)$  in Eq. (160) into a finite part and a pure-pole part. To this purpose, we first consider the kernel  $(1 - K)$  and perform the following formal manipulations:

$$1 - K = 1 - PK - (1 - P)K = [1 - PK(1 - (1 - P)K)^{-1}] [1 - (1 - P)K], \quad (162)$$

which leads to (note the reverse order):

$$\frac{1}{1 - K} = \left[ \frac{1}{1 - (1 - P)K} \right] \left[ \frac{1}{1 - PK(1 - (1 - P)K)^{-1}} \right]. \quad (163)$$

This is the desired factorized expression: the term in the first square bracket is by construction free of collinear singularities, which have been moved to the second

term. Using this factorized form, we can now rewrite the cross-section (159) as

$$\sigma = \bar{M}\bar{\Gamma} = \left\{ H \left[ \frac{1}{1 - (1 - P)K} \right] \right\} \left\{ \left[ \frac{1}{1 - PK(1 - (1 - P)K)^{-1}} \right] \bar{\Gamma} \right\} = M\Gamma, \quad (164)$$

with

$$\begin{aligned} M &\equiv H \left[ \frac{1}{1 - (1 - P)K} \right] \\ \Gamma &\equiv \left[ \frac{1}{1 - PK(1 - (1 - P)K)^{-1}} \right] \bar{\Gamma}. \end{aligned} \quad (165)$$

Now both  $M$  and  $\Gamma$  are finite, hence we can safely let  $\epsilon = 0$  and obtain a well-defined  $d = 4$  result. Note that this subtraction scheme is uniquely defined once the action of  $P$  is specified. The subtraction of the pure pole part (as in Eq. (161)) defines the  $\overline{\text{MS}}$  scheme. In the  $\overline{\text{MS}}$  scheme one chooses to subtract also terms proportional to  $-\gamma_E + \ln 4\pi$  coming from angular integrations. At leading logarithmic accuracy the action of  $P$  is defined as:

$$P_{\overline{\text{MS}}} F = \sum_{k>0} \left( \lim_{\epsilon \rightarrow 0} \epsilon^k F \right) \times \frac{1}{\epsilon^k} \exp [k \epsilon (-\gamma_E + \ln 4\pi)]. \quad (166)$$

We now illustrate this subtraction procedure on the simple case of the  $t$ -channel iteration of Altarelli-Parisi gluons. In this case we have

$$\begin{aligned} K &= \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^{Q^2} \frac{dk_T^2}{(k_T^2)^{1+\epsilon}} \alpha_s (\mu^2)^\epsilon \gamma_0(N) = -\frac{1}{\epsilon} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} (\alpha_s \gamma_0(N)) \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \\ K^2 &= \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^{Q^2} \frac{dk_{T,2}^2}{(k_{T,2}^2)^{1+\epsilon}} (\alpha_s (\mu^2)^\epsilon \gamma_0(N)) \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \times \\ &\times \int_0^{k_{T,2}^2} \frac{dk_{T,1}^2}{(k_{T,1}^2)^{1+\epsilon}} (\alpha_s (\mu^2)^\epsilon \gamma_0(N)) = \frac{1}{2\epsilon^2} \frac{(4\pi)^{2\epsilon}}{\Gamma^2(1-\epsilon)} (\alpha_s \gamma_0(N))^2 \left( \frac{\mu^2}{Q^2} \right)^{2\epsilon} \end{aligned} \quad (167)$$

and so on. Note that  $K^2$  contains both single and double poles:

$$\begin{aligned} \frac{K^2}{\alpha_s^2 \gamma_0^2} &= \frac{1}{2\epsilon^2} + \frac{\ln \left( \frac{\mu^2}{Q^2} \right) + \ln(4\pi) - \gamma_E}{\epsilon} + \left( \ln \left( \frac{\mu^2}{Q^2} \right) + \ln(4\pi) - 2\gamma_E \right) \times \\ &\times \left( \ln \left( \frac{\mu^2}{Q^2} \right) + \ln(4\pi) \right) - \frac{\pi^2}{12} + \gamma_E^2 + O(\epsilon). \end{aligned} \quad (168)$$

Careless subtraction of all poles and  $\ln(4\pi)$ ,  $\gamma_E$  terms would lead to the incorrect result

$$K_{\text{sub}}^2 = \alpha_s^2 \gamma_0^2(N) \ln^2 \frac{Q^2}{\mu^2}. \quad (169)$$

The correct  $\overline{\text{MS}}$  result is instead obtained following the procedure explained above: instead of subtracting poles from  $K^2$  we should consider  $(1-P)[K(1-P)K]$ . In this case we get

$$\begin{aligned} (1-P)K &= (\alpha_s \gamma_0(N)) \left( -\frac{1}{\epsilon} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon + \frac{S_\epsilon}{\epsilon} \right); \\ K(1-P)K &= (\alpha_s \gamma_0(N))^2 \left( \frac{1}{2\epsilon^2} \frac{(4\pi)^{2\epsilon}}{\Gamma^2(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^{2\epsilon} + \right. \\ &\quad \left. - \frac{1}{\epsilon^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon S_\epsilon \right); \end{aligned} \quad (170)$$

$$\begin{aligned} (1-P)[K(1-P)K] &= (\alpha_s \gamma_0(N))^2 \left( \frac{1}{2\epsilon^2} \frac{(4\pi)^{2\epsilon}}{\Gamma^2(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^{2\epsilon} + \right. \\ &\quad \left. - \frac{1}{\epsilon^2} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon S_\epsilon + \frac{1}{2\epsilon^2} S_\epsilon^2 \right) = \\ &= (\alpha_s \gamma_0(N))^2 \frac{1}{2\epsilon^2} \left( \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon - S_\epsilon \right)^2; \end{aligned} \quad (171)$$

with  $S_\epsilon \equiv \exp[\epsilon(\ln(4\pi) - \gamma_E)]$ . Note that this time

$$(1-P)[K(1-P)K] = \frac{1}{2} (\alpha_s \gamma_0)^2 \ln^2 \frac{Q^2}{\mu^2} + O(\epsilon), \quad (172)$$

which is the correct  $\overline{\text{MS}}$  result.

This subtraction procedure can be easily iterated. In particular, with  $n$  kernels the result is

$$\begin{aligned} (1-P)K \dots (1-P)K &= (\alpha_s \gamma_0)^n \frac{(-1)^n}{n!} \frac{1}{\epsilon^n} \left( \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon - S_\epsilon \right)^n = \\ &= (\alpha_s \gamma_0)^n \frac{1}{n!} \ln^n \frac{Q^2}{\mu^2} + O(\epsilon). \end{aligned} \quad (173)$$

Note that before performing the last  $(1-P)$  subtraction, the  $n$ -kernel result contains only a single  $1/\epsilon^n$  pole.

A particularly important feature of this subtraction method is its iterative nature. This is especially well suited for the purpose of this paper, because we can perform all the universal kernel subtractions once for all, and consider separately the single collinear divergence coming from attaching the universal ladder to the process-dependent hard coefficient function, see Eqs. (31), (42). Also, the  $\overline{\text{MS}}$  subtraction factor  $S_\epsilon$  exactly cancels all terms coming from  $(4\pi)^\epsilon/\Gamma(1-\epsilon)$ : the full  $\overline{\text{MS}}$  result can be obtained by just neglecting all  $(4\pi)^\epsilon/\Gamma(1-\epsilon)$  terms and using  $S_\epsilon = 1$ . This simplification only relies on the fact that each emission is accompanied exactly by a  $(4\pi)^\epsilon/\Gamma(1-\epsilon)$  term, which happens if there are no angular correlations. Since in all our derivations angular correlations are subleading, in the main paper we have used this simplified version of the subtraction procedure.

## B Resummation of rapidity distributions in $\overline{\text{MS}}$

In this appendix we provide an explicit derivation of the resummation of the rapidity distribution in the  $\overline{\text{MS}}$  scheme. Because rapidity is determined by the longitudinal momentum components, while collinear singularities appear in transverse momentum integrations, the main features of the  $\overline{\text{MS}}$  derivation are the same of the derivation in Sec. 3.

For simplicity we consider emissions from just one leg. Extension to the full case is straightforward. We start by writing the  $n$ -kernel fully differential cross-section in  $d = 4 - 2\epsilon$  dimensions, see Eq. (41) <sup>6</sup>:

$$\begin{aligned} d\bar{\sigma}^n \left( x, \frac{\mu^2}{Q^2}, \alpha_s, z_i, \xi_i; \epsilon, \right) = \\ P \left( z_n, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \frac{dz_n}{z_n} \frac{d\xi_n}{\xi_n^{1+\epsilon}} \times C(x, z_i, \xi_i, \alpha_s; \epsilon) \times \\ \times P \left( z_{n-1}, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \frac{dz_{n-1}}{z_{n-1}} \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \dots \times P \left( z_1, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \frac{dz_1}{z_1} \frac{d\xi_1}{\xi_1^{1+\epsilon}}, \end{aligned} \quad (174)$$

where as in Eq. (73) the splitting function  $P$  is the inverse Mellin transform of the anomalous dimension  $\gamma$  in Eq. (41). Note that in Eq. (174) we have written  $C$  as the most generic (2PI) function of all the relevant variables. Quasi-collinear kinematics determines the dependence of  $C$  on its arguments. Because rapidity distributions do not factorize in  $N$ -Mellin space, for the time being we write everything in  $x$ -space.

The  $d$ -dimensional rapidity distribution is immediately obtained from the fully differential cross-section Eq. (174):

$$\begin{aligned} \frac{d\bar{\sigma}^n}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s; \epsilon \right) = \\ = \int_x^1 \frac{dz_n}{z_n} P \left( z_n, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \times \dots \times \int_{\frac{x}{z_2 \dots z_n}}^1 \frac{dz_1}{z_1} P \left( z_1, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \times \\ \times \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C_y(x, z_i, \xi_i, y, \alpha_s; \epsilon) \int_0^{\xi_n} \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \dots \times \int_0^{\xi_2} \frac{d\xi_1}{\xi_1^{1+\epsilon}}, \end{aligned} \quad (175)$$

where the rapidity-dependent coefficient function is defined as

$$\begin{aligned} C_y(x, z_i, \xi_i, y, \alpha_s; \epsilon) \equiv \frac{Q^2}{2s z_1 \dots z_n} \int_0^{2\pi} \frac{d\theta}{2\pi} \int d\Pi_{\mathcal{F}} |\overline{\mathcal{M}}(\mathcal{V}(n) + g^*(p_L) \rightarrow \mathcal{F})|^2 \times \\ \times \delta_4(P_I - P_f) \times \delta \left( y - \frac{1}{2} \ln \frac{E_S + p_{Sz}}{E_S - p_{Sz}} \right), \end{aligned} \quad (176)$$

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<sup>6</sup>Here again we omit all  $(4\pi)^\epsilon/\Gamma(1-\epsilon)$  terms in view of the  $\overline{\text{MS}}$  subtraction, as discussed in Appendix A.

in terms of the spin and color averaged squared amplitude  $|\overline{\mathcal{M}}|^2$  for the off-shell process  $\mathcal{V}(n) + g^*(p_L) \rightarrow \mathcal{F}$  (see Fig. 2), and the polarization sum for the off-shell gluon is performed through the projector  $\mathcal{P}$  Eq. (12). We now use the fact that we are in quasi-collinear kinematics (i.e.  $\xi_i \ll \xi_{i+1}$ ) and in the small  $x$  regime Eq. (8). This implies first, that  $C_y$  is sensitive only to the largest transverse momenta  $\xi_n$  and second, that up to terms which vanish when  $k_T^2 \ll s$  the effect of the ladder insertion to  $C_y$  is just a longitudinal boost. We have thus

$$\begin{aligned} C_y(x, z_i, \xi_i, y, \alpha_s; \epsilon) &= \\ &= C_y\left(\frac{x}{z_1 \dots z_n}, \xi_n, y - \frac{1}{2} \ln z_1 - \dots - \frac{1}{2} \ln z_n, \alpha_s; \epsilon\right) + O\left(\frac{k_T^2}{s}\right). \end{aligned} \quad (177)$$

Since upon integration  $O(k_T^2/s)$  terms give rise to contributions suppressed by powers of  $x$ , we can safely discard them.

Thanks to quasi-collinear kinematics we can thus write Eq. (175) as

$$\begin{aligned} \frac{d\bar{\sigma}^n}{dy}\left(x, \frac{\mu^2}{Q^2}, y, \alpha_s; \epsilon\right) &= \\ &= \int_x^1 \frac{dz_n}{z_n} P\left(z_n, \left(\frac{\mu^2}{Q^2}\right)^\epsilon, \alpha_s; \epsilon\right) \times \dots \times \int_{\frac{x}{z_2 \dots z_n}}^1 \frac{dz_1}{z_1} P\left(z_1, \left(\frac{\mu^2}{Q^2}\right)^\epsilon, \alpha_s; \epsilon\right) \times \\ &\times \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C_y\left(\frac{x}{z_1 \dots z_n}, \xi_n, y - \frac{1}{2} \ln z_1 - \dots - \frac{1}{2} \ln z_n, \alpha_s; \epsilon\right) \times \\ &\times \int_0^{\xi_n} \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \dots \times \int_0^{\xi_2} \frac{d\xi_1}{\xi_1^{1+\epsilon}} + O(x). \end{aligned} \quad (178)$$

From now on we will systematically omit all  $O(x)$  terms.

Convolution products in Eq. (178) turn into ordinary ones by Fourier-Mellin transformation:

$$\begin{aligned} \frac{d\bar{\sigma}^n}{dy}\left(N, \frac{\mu^2}{Q^2}, b, \alpha_s; \epsilon\right) &= \left[\gamma\left(N - \frac{ib}{2}, \left(\frac{\mu^2}{Q^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \times \\ &\times \int_0^\infty \frac{d\xi_n}{\xi_n^{1+\epsilon}} C_y(N, \xi_n, b, \alpha_s; \epsilon) \times \int_0^{\xi_n} \left[\gamma\left(N - \frac{ib}{2}, \left(\frac{\mu^2}{Q^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\xi_{n-1}}{\xi_{n-1}^{1+\epsilon}} \times \\ &\times \dots \times \int_0^{\xi_2} \left[\gamma\left(N - \frac{ib}{2}, \left(\frac{\mu^2}{Q^2}\right)^\epsilon, \alpha_s; \epsilon\right)\right] \frac{d\xi_1}{\xi_1^{1+\epsilon}}. \end{aligned} \quad (179)$$

This expression is identical to Eq. (41), but with the anomalous dimension  $\gamma$  is evaluated at  $N - ib/2$ . This does not interfere with collinear singularities, hence the



$\overline{\text{MS}}$  subtraction is performed as in the inclusive case, with the result (see Eq. (44))

$$\begin{aligned} \frac{d\sigma}{dy} &= \sum_{n=1}^{\infty} \frac{d\bar{\sigma}^n}{dy} = \gamma \left( N - \frac{ib}{2}, \left( \frac{\mu^2}{Q^2} \right)^\epsilon, \alpha_s; \epsilon \right) \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C_y(N, \xi, b, \alpha_s; \epsilon) \times \\ &\times \exp \left[ \frac{1}{\epsilon} \sum_k \frac{\tilde{\gamma}_k(N - \frac{ib}{2}, \alpha_s; 0)}{k} \left( 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^{k\epsilon} \frac{\tilde{\gamma}_k(N - ib/2, \alpha_s; \epsilon)}{\tilde{\gamma}_k(N - ib/2, \alpha_s; 0)} \right) \right], \end{aligned} \quad (180)$$

where  $\tilde{\gamma}_k$  was defined in Eq. (45). As in the inclusive case, Eq. (180) has a finite  $\epsilon \rightarrow 0$  limit:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \quad & \frac{1}{\xi^{1+\epsilon}} \exp \left[ \frac{1}{\epsilon} \sum_i \frac{\bar{\alpha}_s^i}{i} \gamma_i \left( N - \frac{ib}{2}, 0 \right) \left( 1 - \left( \frac{\mu^2}{Q^2 \xi} \right)^{i\epsilon} \frac{\gamma_i(N - ib/2, \epsilon)}{\gamma_i(N - ib/2, 0)} \right) \right] = \\ &= \xi^{\gamma(N - \frac{ib}{2}, \alpha_s) - 1} \exp \left[ \gamma \left( N - \frac{ib}{2}, \alpha_s \right) \ln \frac{Q^2}{\mu^2} \right] e^{-\sum_i \frac{\alpha_s^i \gamma_i(N - ib/2)}{i}} = \\ &= \xi^{\gamma(N - \frac{ib}{2}, \alpha_s) - 1} \exp \left[ \gamma \left( N - \frac{ib}{2}, \alpha_s \right) \ln \frac{Q^2}{\mu^2} \right] \mathcal{R} \left( N - \frac{ib}{2}, \alpha_s \right). \end{aligned} \quad (181)$$

Using Eq. (181) the resummed result immediately follows (with  $\mu^2 = Q^2$ , thus omitting the explicit dependence on  $\mu^2/Q^2$ ):

$$\begin{aligned} \frac{d\sigma}{dy}(N, \alpha_s(Q^2), b) &= \gamma \left( N - \frac{ib}{2}, \alpha_s(Q^2) \right) \times \\ &\times \int_0^\infty d\xi \xi^{\gamma(N - ib/2, \alpha_s(Q^2)) - 1} C(N, \xi, \alpha_s(Q^2), b) \mathcal{R} \left( N - \frac{ib}{2}, \alpha_s(Q^2) \right), \end{aligned} \quad (182)$$

which is the same of Eq.(47) but this time with a shifted  $\mathcal{R}$  factor. Note that at LLx we also have

$$\gamma \left( N - \frac{ib}{2}, \alpha_s \right) = \gamma \left( \frac{\alpha_s}{N - ib/2} \right); \quad \mathcal{R} \left( N - \frac{ib}{2}, \alpha_s \right) = \mathcal{R} \left( \frac{\alpha_s}{N - ib/2} \right). \quad (183)$$

We finally show how to deal with a scheme change in rapidity distribution, thereby in particular explaining the shift in the argument of  $\mathcal{R}$  in Eq. (182). To this purpose, assume a scheme change from  $\overline{\text{MS}}$  to some “small  $x$ ” scheme is performed by dividing the parton distribution by a LLx function  $\mathcal{N}$ :

$$M_{\text{small } x} \equiv M_{\overline{\text{MS}}} \mathcal{N}; \quad \Gamma_{\text{small } x} \equiv \frac{1}{\mathcal{N}} \Gamma_{\overline{\text{MS}}}. \quad (184)$$

Using the  $\overline{\text{MS}}$  kernel expansion Eq. (165) we can rewrite  $M_{\text{small } x}$  as

$$M_{\text{small } x} = H \frac{1}{1 - (1 - P)K} \mathcal{N} = H [1 + (1 - P)K + (1 - P)[K(1 - P)K] + \dots] \mathcal{N}. \quad (185)$$

where (see Appendix A) all products must be understood as convolution in  $x$  and  $k_T$  spaces. It follows that the effect of the scheme change is the same as that of a kernel insertion on the first (outer) rung of the ladder:

$$\begin{aligned} \frac{d\bar{\sigma}^1}{dy} \left( x, \frac{\mu^2}{Q^2}, y, \alpha_s; \epsilon \right) &= \int_x^1 \frac{dz}{z} \mathcal{N}(z, \alpha_s; \epsilon) \times \\ &\times \int_{x/z} \frac{dz_1}{z_1} P \left( z_1, \alpha_s, \left( \frac{\mu^2}{Q^2} \right)^\epsilon; \epsilon \right) \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C \left( \frac{x}{zz_1}, \xi, y - \frac{1}{2} \ln z - \frac{1}{2} \ln z_1, \alpha_s; \epsilon \right). \end{aligned} \quad (186)$$

In Fourier-Mellin space the convolutions in Eq. (186) turn into ordinary products

$$\begin{aligned} \frac{d\bar{\sigma}^1}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s; \epsilon \right) &= \mathcal{N} \left( N - \frac{ib}{2}, \alpha_s; \epsilon \right) \times \\ &\times \gamma \left( N - \frac{ib}{2}, \alpha_s, \left( \frac{\mu^2}{Q^2} \right)^\epsilon; \epsilon \right) \int_0^\infty \frac{d\xi}{\xi^{1+\epsilon}} C_y \left( N - \frac{ib}{2}, b, \alpha_s; \epsilon \right), \end{aligned} \quad (187)$$

so the argument of  $\mathcal{N}$  ends up being also shifted when used for computing rapidity distributions. Clearly this holds also if we consider the full ladder and not only a single kernel insertion.

Collecting the results of this section, we can write the full  $\overline{\text{MS}}$  rapidity distribution resummation formula as

$$\begin{aligned} \frac{d\sigma}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s(\mu^2) \right) &= \left[ \gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) \exp \left[ \gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) \ln \frac{Q^2}{\mu^2} \right] \times \right. \\ &\times \left. \int_0^\infty d\xi \xi^{\gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) - 1} C \left( N, \xi, b, \alpha_s(\mu^2) \right) \right] R \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right). \end{aligned} \quad (188)$$

The extension of Eq. (188) to emissions from both legs along the lines of Sec. 3 is straightforward

$$\begin{aligned} \frac{d\sigma}{dy} \left( N, \frac{\mu^2}{Q^2}, b, \alpha_s(\mu^2) \right) &= \gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) \exp \left[ \gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) \ln \frac{Q^2}{\mu^2} \right] \times \\ &\times \gamma \left( \frac{\alpha_s(\mu^2)}{N + \frac{ib}{2}} \right) \exp \left[ \gamma \left( \frac{\alpha_s(\mu^2)}{N + \frac{ib}{2}} \right) \ln \frac{Q^2}{\mu^2} \right] \times R \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) R \left( \frac{\alpha_s(\mu^2)}{N + \frac{ib}{2}} \right) \times \\ &\times \int_0^\infty d\xi \xi^{\gamma \left( \frac{\alpha_s(\mu^2)}{N - \frac{ib}{2}} \right) - 1} \int_0^\infty d\bar{\xi} \bar{\xi}^{\gamma \left( \frac{\alpha_s(\mu^2)}{N + \frac{ib}{2}} \right) - 1} C \left( N, \xi, \bar{\xi}, b, \alpha_s(\mu^2) \right). \end{aligned} \quad (189)$$

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